The Best-Shot All-Pay (Group) Auction with Complete Information*

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Abstract

We analyze an all-pay group contest problem in which individual members' efforts are aggregated via the best-shot technology and the prize is a public good for the winning group. The interplay of within-group free-riding and across-group competition allows for a wide variety of equilibria, according to how well groups are able to overcome internal free-riding. In a symmetric model we derive equilibria in which multiple agents per group are active, in contrast with the existing literature. Our findings differ qualitatively from those of the individualistic all-pay auction: rents are not necessarily dissipated in equilibrium, total expected efforts vary across equilibria, and participation is expected to be greater. Moreover, equilibria with greater symmetry of behavior within a group are shown to have more "wasted" effort but also greater payoffs as overall efforts are lower. In contrast to many results in the literature, free-riding can be beneficial for players as it reduces competition among groups. Examples of asymmetric group contests are also studied.

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1 Introduction

Group conflict is a common experience of daily life. For example, the results of sports team competitions are often front page news, and so are the outcomes of elections contested by coalitions of multiple parties. In these and many other instances, individual group members make costly contributions that improve the chances of their group to succeed in the across-group struggle. And all members of the successful group may receive a benefit that is independent of the individual effort actually expended. Thus, a player faces a tradeoff: he has an incentive to rely on his teammates' efforts and lessen his own, but doing so may reduce his team's chance of winning the contest.

There are many different ways to model group conflict; interesting facets include the nature of the prize, the rule that is used to determine the winner, and the way individual efforts are transformed into the group's effort (sometimes referred to as group effort technology). We make three key assumptions in our analysis. *First*, we consider a pure public good structure for the prize, so that all members of the winning coalition benefit from victory regardless of their individual efforts, and the trophy is consumed nonrivalrously and nonexclusively by all members of the winning team.[1] *Second*, we identify a group's effort, as a function of the vector of individual members' efforts, through the best-shot aggregator function (Hirschleifer, 1983). This is in contrast with the most common assumption of the summation aggregation function.[2] *Third*, to determine the outcome of group competition we consider an all-pay auction setup: the coalition with the largest group effort is the winner, and efforts are expended by both victors and vanquished.[3]

Our paper is most closely related to Chowdhury *et al.* (2013a), as we share the first two assumptions above. Thus, many of the examples they provide (*e.g.*, Tour de France, Formula 1, competing defense coalitions such as NATO and the Warsaw Pact) apply for our situation as well. Our paper is also closely related to Baik *et al.* (2001) and Chowdhury *et al.* (2013b), since we share all assumptions but the form of the aggregator function. There is a close relationship between that contest of Baik *et al.* (2001) and the individualistic all-pay auction of Baye *et al.* (1996), since, notably, Baik *et al.* find that in equilibrium only one player is active in each group. Therefore, our paper is closely related to Baye *et al.* (1996) as well, the

[1]This specification is widely used in the literature (see for instance Baik *et al.*, 2001; Baik, 2008; Chowdhury *et al.*, 2013a; and Katz *et al.*, 1990). Another line in the literature models the prize as a private good and then focuses on allocation of that prize among the members of the winning group (see, for example, Baik and Lee (2001, 2007), Baik and Lee (2012), Katz and Tokatlidu, 1996; Lee, 1995; Münster, 2007; Nitzan, 1991a, 1991b; and Wärneryd, 1998). Finally, one may consider prizes that are combinations of private and public goods as in Estban and Ray (2001) and Nitzan and Ueda (2011), for instance.

[2]See, for instance, Baik *et al.* (2001), Baik (2008), Esteban and Ray (2001), Katz *et al.* (1990), and Nitzan (1991a). Some recent attempts accounting for departures from perfect substitutability of efforts include Lee (2012), Chowdhury *et al.* (2013a), Chowdhury and Topolyan (2013), and Kolmar and Rommeswinkel (2013), who consider a CES aggregator function in which efforts range from perfect complements (weakest-link) to perfect substitutes (summation).

[3]Among others, Baik *et al.* (2001) study such a "perfectly discriminating" contest, using the terminology of Nitzan (1991a). Our assumption is in contrast with numerous studies adopting a lottery contest success function, which was introduced by Tullock (1980). According to this methodology the group that exerts the highest effort is not guaranteed to win the prize, but higher effort increases its chances of victory. Among these studies are Baik (2008), Chowdhury *et al.* (2013a), Katz *et al.* (1990), Münster (2009), Kolmar and Rommeswinkel (2013), and Dijkstra (1998), to mention just a few.

only difference being that we consider group competition rather than individualistic competition.

On the one hand, therefore, a narrow view of our contribution is technical, as we fill in a gap in the literature on group all-pay auctions. On the other hand, and more importantly, with the best-shot aggregator we believe we offer a framework especially suited to describe competition in which groups bring forth arguments or ideas, rather than aggregate monetary contributions. For example, the institution of *amicus curiae* (AC) in the U.S. legal system illustrates the kind of group competition we have in mind.

ACs, *i.e.*, "friends of the court," are parties other than the petitioner or the respondent who add arguments to legal proceedings to bolster the case of one of the two contesting sides. AC briefs are typically employed at the appellate level (the federal Supreme Court and many state Supreme Courts), and cases of extreme social importance attract many AC briefs.[4] Clearly, an AC brief that simply repeats the ideas of one of the contesting parties adds very little to the decision-making process of justices. Because of this, a summation-of-effort technology (as in Baik *et al.*, 2001) may be less appropriate than the best shot to capture the contribution of an individual AC brief to the success of the group. And all interested parties benefit from a favorable decision, regardless of the effort they expended, thus justifying our assumption that the prize is a public good within a group. While Chowdhury *et al.* (2013a) also consider a best-shot aggregation technology, in their equilibria only one agent per group takes action. In contrast, we exhibit equilibria in which multiple agents per group exert effort; therefore our setup can account for multiple AC briefs being filed on one (or both) sides.

Because of the within-group public good nature of prize, our paper is related to the literature on the private provision of public goods (PPPG) and the volunteer's dilemma (VD). Free-riding is often predicted in PPPGs (see for example Olson, 1965; Palfrey and Rosenthal, 1984; Bergstrom *et al.*, 1986; Vicary, 1997; Xu, 2001; and Barbieri and Malueg, 2008a, 2008b, and 2012). The VD is a situation where the discrete effort of just one player is enough to benefit all members of the group. The best-shot PPPG can therefore be seen as a VD with continuous effort. It is commonly documented that in symmetric equilibria of VDs and best-shot PPPGs the probability that a public good is produced (or its quantity) decreases as the group becomes larger (Diekmann, 1984; Harrington, 2001; and Barbieri and Malueg, 2012). Thus, the free-riding problem becomes so severe that the expected value of the public good provided decreases despite the fact that there are more contributors. Indeed, we find a similar effect in our contest setup.

Finally, our work is relevant to the literature on contests with identity-dependent externalities, or IDE (*e.g.*, Funk, 1996; Das Varma, 2002; and Klose and Kovenock, 2012 and 2013). In such contests, a player who loses may care about who has won, i.e., the payoff of a loser may depend on the identity of the winner.

[4]For instance, in the 2012 U.S. Supreme Court Decision on the Affordable Care Act, over 150 AC briefs were filed on both sides.

Thus, the situation where a player is equally happy if anyone else "in her group" wins the all-pay auction is an example of an all-pay auction with IDE. And such a situation is precisely the best-shot all-pay auction we study. Because of the stronger structure we impose, our results considerably sharpen (for our more restrictive setup) the insights gleaned from the examples in Klose and Kovenock (2013), for instance.

We find a variety of equilibria where more than one agent within a group is active (in some equilibria all players are active). This is an interesting finding, especially because many papers in the contest literature focus on equilibria where only one player in each group is active (Baik *et al.*, 2001; Baik, 2008; and Chowdhury *et al.*, 2013a). Beyond adding to the realism of equilibrium predictions, the outcomes we find are qualitatively different from those of the standard, individualistic all-pay auction (Baye *et al.*, 1996), which is what arises if there is only one active agent per group. Three major differences stand out.

First, rents in symmetric contests might not be completely dissipated, even for active agents in our perfectly discriminating setup.[5] Rents are not completely dissipated when at least two players in a group are active. Rent dissipation, however, is complete if only one player in each group is active (which resonates with the individualistic all-pay auction and the group all-pay auction setting analyzed by Baik *et al.*, 2001). Also, regardless of the number of active players per group, as the number of groups approaches infinity the rent dissipation among active players becomes complete. *Second*, payoffs may vary across equilibria and identical players may earn different payoffs.[6] This observation is in contrast with the remarkable results regarding the payoff equivalence of equilibria of individual all-pay auctions (Baye *et al.*, 1996) and generic contests (Siegel, 2009). Further, the sum of expected efforts is not constant across different equilibria of a symmetric contest (in contrast to Baik *et al.*, 2001; Baik, 2008; and Topolyan, 2013). *Third*, in asymmetric contests, a wider participation is possible.

The reason behind these differences is that a wide variety of equilibria exist in our setup, according to how well group members are able to cope with the problem of internal free-riding. Indeed, there exist equilibria in which group members are unable to overcome the internal free riding problem, group efforts are low on average, and therefore, if free-riding afflicts all groups, group members fare *well*, since competition across groups is low. And if groups are better able to overcome internal free-riding tendencies, then across-group competition becomes fiercer and equilibrium payoffs decrease, up to the point at which rents are completely dissipated and some groups may remain completely inactive.

We identify two sources for the incentive to free-ride. The first is the number of active agents: because of the link with VDs and best-shot PPPGs previously described, a broader participation within each active group creates a larger incentive to free-ride among active agents and therefore generates a smaller rent dis-

[5] Rent dissipation is typically not complete in imperfectly discriminating contests, *e.g.*, Nitzan (1991a), Baik (2008), and Chowdhury *et al.*(2013a).

[6] A similar observation was made by Klose and Kovenock (2013) regarding the all-pay auctions with IDE's.

sipation, a larger payoff for active agents, and, in asymmetric contests, a smaller barrier to the participation of groups with lower values for the public good.

The second reason an agent has to shade his contribution is the possible wasteful duplication of effort within a group. We offer a measure of this waste and provide a family of equilibria where active players within the same group randomize over different intervals; in particular, active teammates' strategies have different supports composed of interlaced intervals.[7] The finer the interlacing, the larger the probability of wasted contribution, the larger the incentive to shade one's contribution, again with the consequence that overall efforts are reduced.

In comparison to other group contests, the most striking difference brought about by the best-shot technology we employ is a return of the "group size paradox": in semi-symmetric equilibria with asymmetric groups the effect of group size on the group's probability of winning is negative (i.e., a larger group is less likely to win, other things equal).[8] Our result is a counterpoint to Katz *et al.*, (1990), Riaz *et al.*, (1995), Esteban and Ray (2001), and Nitzan and Ueda (2011).[9] The main reason for this difference is the strong within-group free riding above described.

The rest of the paper is organized as follows. Section 2 describes the basic characteristic of the model. Section 3 analyzes symmetric contests, exploring both symmetric equilibria and equilibria in which group efforts are symmetric, but individual active agents' efforts are not. Section 4 analyzes asymmetric contests, with special attention to two- and three-group settings. Section 5 concludes.

2 All-pay auction contests with best-shot group performance

We model a situation in which several groups compete for a prize that is a public good for the winning group. This eliminates concerns about how to share a prize in the event of winning; however, it introduces free riding among members in the group as one can benefit from a win even when putting forth relatively little effort. Now individuals must weigh the benefits of free riding on teammates against the risk of losing out altogether to another team that provides greater effort. When there are differences among teams, we model those as differences in *values* for the good.

We envision N groups possibly competing for the prize, with group i having n_i members, $i = 1, \dots, N$. Let $\underline{n} \equiv \min\{n_1, \dots, n_N\}$ denote the size of the smallest group. We index members of group i by $i_1, i_2, \dots, i_{n_i}$,

[7] Within-group asymmetric behavior of active teammates has not been a common object of interest in the contest literature; see however Topolyan (2013) for another such example. To the best of our knowledge, interlaced-interval supports of equilibrium strategies are an entirely new result in the literature on group contests. Even those studies that document a continuum of equilibria (*e.g.*, Lee, 2012) find that in any equilibrium the supports of active players are the same.

[8] See also the group all-pay auctions with a weakest-link impact function in Chowdhury *et al.*, (2013b).

[9] Nitzan and Ueda (2011) demonstrate that endogenous group formation militates against the group size paradox. We do not consider endogenous group formation here.

$i = 1, \ldots, N$. We assume the constant marginal cost of effort common to all players is $c > 0$ which, without loss of generality, we normalize to $c = 1$. We let $v_i > 0$ denote the common gross benefit of winning to members of group i. A member of group i exerting effort x then earns final payoff $v_i - x$ if group i wins and $-x$ if it does not.

The groups compete in an all-pay contest where each team is judged according to its best effort ("best shot"). So the performance of group i is given by $X_i = \max\{x_{i_1}, \ldots, x_{i_{n_i}}\}$. Group i wins if $X_i > X_j$, for all $j \neq i$; in the case that k groups tie for maximum performance, each of the tying groups has probability $1/k$ of being designated the winner.

We say that a player is *active* if he exerts a strictly positive effort with strictly positive probability, and an *active* group is one with at least one active player.

3 Symmetric active groups

As a starting point we further assume the value of winning is common across groups, so $v_i = v$ for all i. We begin by analyzing the case where each team has a common number of active players, and then, for the case of two teams, we allow active groups to differ in size.

3.1 Semi-symmetric equilibria

Here we additionally suppose that each group has a common number of active players. Such would be typical, for example, of most sporting events. We construct semi-symmetric equilibria in which all players' equilibrium payoffs are positive. (We say *semi*-symmetric because, while active players all use the same strategy, in equilibrium some players may remain inactive.) In particular, we show that for any $m \in \{1, \ldots, \underline{n}\}$, there is an equilibrium in which exactly m players on each team are active (the others are "extreme" free riders) and if $m > 1$, then all players' payoffs are positive. We look for a semi-symmetric mixed-strategy equilibrium, where each active player exerts effort according to the cumulative distribution function (cdf) F. As usual, F does not admit any atoms of probability and its support has lower limit 0; the upper limit will be determined as part of the equilibrium.[10] Thus, F is continuous, which in turn implies the equilibrium chance of ties is zero.

Suppose $m \in \{1, \ldots, \underline{n}\}$ players on each team choose effort according to cdf F. We now derive the cdf that just makes these active players willing to follow this randomization. We will then show that no inactive

[10] To see that F does not admit any atoms of probability, suppose to the contrary that an atom exists at level $\hat{x}$. This means that $\hat{x}$ maximizes utility against F. But an increase in a player's contribution from $\hat{x}$ to $\hat{x}+\varepsilon$ discretely increases the probability that the player's team wins the auction, with only a marginal increase in cost. Therefore, for ε small, a contribution of $\hat{x}+\varepsilon$ is a profitable deviation, contradicting the assumption that F was an equilibrium strategy.

player would choose to become active. Consider the problem from the point of view of an active player on team 1, say, player 1_1. The cdf of the maximum of all efforts other than player 1_1's is $(F)^{Nm-1}$. Team 1 can win in either of two ways, given that player 1_1's contribution is x: either player 1_1's effort is the global best shot or the global best shot is larger than x and is achieved by someone on team 1. The first event happens with probability $(F(x))^{Nm-1}$, the second with probability $\left(1-(F(x))^{Nm-1}\right) \times (m-1)/(Nm-1)$. Therefore, player 1_1's expected payoff is

$$\begin{aligned} V(x) &= -x + v\left[[F(x)]^{Nm-1} + \frac{m-1}{Nm-1}\left(1-[F(x)]^{Nm-1}\right)\right] \\ &= -x + v\left[\frac{m-1}{Nm-1} + \frac{Nm-m}{Nm-1}[F(x)]^{Nm-1}\right]. \end{aligned} \tag{1}$$

Since F is continuous, V is continuous, which implies that the support of F is an interval $[0, \bar{x}]$, for some $\bar{x} > 0$.[11] The absence of atoms in the equilibrium strategy implies for an active player the payoff is $V(0) = \left(\frac{m-1}{Nm-1}\right) v$, so, from (1), indifference over the randomization interval implies

$$F(x) = \left[\left(\frac{Nm-1}{Nm-m}\right)\frac{x}{v}\right]^{\frac{1}{Nm-1}}, \qquad \text{with} \qquad \bar{x} = \left(\frac{Nm-m}{Nm-1}\right)v. \tag{2}$$

The above analysis shows that indeed each active player can do no better than to use this cdf F. To complete the verification of the equilibrium we must show that no inactive players would wish to become active. We do so in the Appendix and obtain the following result.

Proposition 1 (Equilibrium)**.** *For each $m \in \{1, \ldots, \underline{n}\}$ there exists an equilibrium in the all-pay auction where m players on each team independently contribute effort according to the cdf in* (2) *and all others exert no effort. Each team wins with probability $1/N$, and expected payoffs are v/N to each inactive player and $\frac{m-1}{Nm-1}v$ to each active player. Given m active players per team,* (2) *describes the unique semi-symmetric equilibrium strategy.*

Proposition 1 reveals a first important contrast with the symmetric individualistic all-pay auction. Baye *et al.* (1996) found that all players earn a payoff of zero in all equilibria. In contrast, we find in the semi-symmetric equilibrium with $m \geq 2$, all players earn strictly positive payoffs. It is the possibility of winning based on teammates' efforts that ensures even an active player a positive equilibrium payoff, leading to efforts surely less aggressive than v. Moreover, there are equilibria in which all teams have a common

[11] To see this, suppose to the contrary that there exists an interval (x^l, x^h), with $0 \leq x^l < x^h \leq \bar{x}$, in which no contributions fall, and x^l and x^h are in the support of F. In equilibrium it must be that $V(x) = v^*$ a.e.-F, and because V is continuous it follows that $V(x) = v^*$ on the support of F. Therefore, $V(x^l) = V(x^h)$, which, because F is constant on $[x^l, x^h]$, is easily seen from (1) to be impossible.

number of active players, which may be as many as the number of members on the smallest team.

We now further pursue the effects of having multiple active players within a group by considering the comparative statics with respect to m. It is straightforward to see that as m increases, the equilibrium strategy F in (2) shifts leftward in the sense of first-order stochastic dominance. Moreover, even as the number of active players on each team increases, the distribution of a team's best shot shifts leftward in the sense of first-order stochastic dominance.[12] A team's total expected effort is

$$m \int_0^{\bar{x}} y \, dF(y) = \frac{mv}{N} \times \frac{N-1}{Nm-1}, \tag{4}$$

which decreases monotonically toward $\frac{N-1}{N^2}v$ as m increases. Most interestingly, as active groups symmetrically get larger the total expected benefit to active members becomes arbitrarily large while the team's total expected cost remains bounded, implying that "rent dissipation" effectively becomes insignificant. Indeed, from (4) we see that the ratio of a team's expected effort to its expected gross benefit of active members (mv/N) is $(N-1)/(Nm-1)$, which monotonically decreases in m with limit 0. Not surprisingly, then, we find that groups prefer equilibria with a greater number of active players. Moving from m to $m+1$ active players per group has no effect on payoffs of players who remain inactive. A player who continues to be active earns a larger payoff if the number of active players per group increases (because $\bar{x}$ decreases with m), but the player switching from inactive to active suffers a decrease in payoff. Overall, because an increase in m leaves all groups' chances of winning at $1/N$ and expected efforts are lower, each group's collective payoff increases as the number of active players in a semi-symmetric equilibrium increases.

Intuitively, then, the effect of an increase in m is to increase the incentive to free ride for active members, because agents can count on more teammates to take action. Therefore, groups' efforts decrease, just as happens in VDs and best-shot PPPGs. In contrast to what happens there, though, in our case more free-riding is beneficial for players, because it reduces competition among groups.

We further analyze competition among groups by considering the effects of increasing N. In the individualistic all-pay auction equilibrium ($m=1$), increases in N cause a FOSD shift to lower efforts. In contrast, when $m \geq 2$ the shift is not fully in accord with FOSD. This follows because, while for lower efforts, increases in N do increase F, at the same time $\bar{x}$ increases, meaning that maximum possible efforts are larger

[12] To see this, observe from (2) that the distribution of the best shot is

$$(F(x))^m = \left[\left(\frac{Nm-1}{Nm-m}\right)\frac{x}{v}\right]^{\frac{m}{Nm-1}} = \left[\left(\frac{N-1/m}{N-1}\right)\frac{x}{v}\right]^{\frac{1}{N-1/m}} = \left(\frac{zy}{N-1}\right)^{\frac{1}{z}}, \tag{3}$$

where $y = x/v$, $z = N - 1/m$, and $zy < N-1$. Over the range considered, the last expression in (3) is clearly increasing in z, which itself is an increasing function of m.

Additionally, as $m \to \infty$ (or $N \to \infty$), an active player's strategy converges in probability to a unit mass at 0 (even as the upper limit of the distribution's support remains bounded away from 0). Nevertheless, holding N fixed, the distribution of a team's best shot converges to a nondegenerate distribution as $m \to \infty$.

as N increases. As the number of teams becomes arbitrarily large, the distribution of a team's maximum effort, $(F(x))^m$, converges in probability to a unit mass at 0, even as the distribution of the overall best shot converges to the uniform distribution on $[0, v]$.

Finally, if $N > 2$, then there are equilibria in which some *teams* are inactive; however, the possibilities are limited, as shown next.

Proposition 2 (The possibility of inactive teams)**.** *Consider a semi-symmetric equilibrium in which each active group has m active players. If $m = 1$, then there may be as many as $N - 2$ inactive teams. If $m \geq 2$, then all teams are active.*

It is trivial to see that any equilibrium has at least two active teams. For $m = 1$, Proposition 2 follows from the analysis of Baye *et al.* (1996) for the individualistic all-pay auction. If, however, $m \geq 2$, then it must be that all teams are active, for when $m \geq 2$, the largest effort of any active team member will be less than v (cf. (2)), in which case a member of an inactive team could raise her payoff above zero by exerting effort $\bar{x}$.[13]

One may view Proposition 2 as identifying another difference with the individualistic all-pay auction: participation is encouraged by having teams with multiple active members. While Baye *et al.* (1996) found that the number of active players in the symmetric individualistic all-pay auction is at least 2 and *may* be more, we find that in equilibria with at least two active players per group, all groups *must* be active. This expansion of participation will be reinforced in the analysis of asymmetric groups in Section 4.

3.2 Asymmetric equilibria

Here we explore the possibility that equilibria may be asymmetric even though the active groups are symmetric. For simplicity only, we focus on the case of two active groups, each with two active players. We designate the teams as 1 and 2, and we suppose each group has a member a and a member b. Baye *et al.* (1996) found in the all-pay auction with complete information that even when all players have common value v for winning, there exist equilibria beyond the one in which all active players use the same strategy. In particular, they find equilibria where two players randomize over $[0, v]$ and other players randomize over sets of the form $\{0\} \cup [b_i, v]$, where b_i is a free player-specific parameter. However, such strategies cannot be part of an equilibrium here.[14]

[13] The reader may have noticed this argument would apply even to asymmetric equilibria as long as at least one group has at least 2 active members. Because each active member will have a strictly positive payoff, it must be that $\bar{x} < v$.

[14] To see this, consider group 1. Suppose player a uses strategy F_a, player b – strategy F_b. Let H denote the cdf of the best-shot of group 2. The maximum element of the support of the best shot of each group must be the same. It must be that H has interval support $[0, \bar{x}]$, for suppose instead that the support of H has a "hole," say (x_a, x_b), where $0 \leq x_a < x_b < \bar{x}$ Then group-1 players would choose no efforts in (x_a, x_b), so a group-2 player exerting effort in a small neighborhood $(x_b, x_b + \varepsilon)$ could slightly reduce these efforts, thereby reducing his cost without reducing his team's probability of winning, contradicting

The foregoing discussion leads us to seek equilibria where active teammates' strategies have (essentially) nonoverlapping supports.

Example 1 (Symmetry across teams, asymmetry within a team)**.**

We look for an equilibrium where the a players on each team use strategy F_a with support $[0, x_m]$ and the b players use strategy F_b with support $\{0\} \cup [x_m, \bar{x}]$. Here $F_b(x_m) = F_b(0)$ denotes the probability with which a b player chooses effort 0. Also, $H(x) = F_a(x)F_b(x)$ is the cdf of a team's best shot.

First consider group 1's player b. The player's payoff to zero effort is

$$\begin{aligned} V_b(0) &= v \int_0^{x_m} H(y)\, dF_a(y) \\ &= v \int_0^{x_m} F_b(x_m)F_a(y)\, dF_a(y) \\ &= \frac{v}{2}F_b(x_m) \int_0^{x_m} \frac{d}{dy}\left(F_a^2(y)\right)\, dy \\ &= \frac{v}{2}F_b(x_m), \end{aligned} \tag{6}$$

where the final equality uses $F_a(0) = 0$ and $F_a(x_m) = 1$. Here (6) establishes the payoff that player b is also to obtain from efforts in $[x_m, \bar{x}]$. In particular, for effort x_m the player's payoff then satisfies

$$\frac{v}{2}F_b(x_m) = V_b(x_m) = -x_m + vH(x_m) = -x_m + vF_b(x_m),$$

which implies

$$x_m = \frac{v}{2}F_b(x_m). \tag{7}$$

Now for $x \in (x_m, \bar{x}]$, the player's payoff must satisfy $\frac{v}{2}F_b(x_m) = V_b(x) = -x + vH(x) = -x + vF_b(x)$, from which, by (7), we obtain

$$F_b(x) = \frac{x_m + x}{v} \qquad \forall x \in [x_m, \bar{x}]. \tag{8}$$

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the assumption that the initial strategies constituted an equilibrium.

Now for group 1, the payoff to player a using effort x is

$$V_{1a}(x) = -x + v\left(1 - \int_x^{\bar{x}} F_b(y)\, dH(y)\right);$$

if $x > 0$ is a point of increase of F_a, then

$$0 = V'_{1a}(x) = -1 + vF_b(x)\, dH(x). \tag{5}$$

Similarly for player b: if z is a point of increase of F_b, then $0 = -1 + vF_a(z)\, dH(z)$. Consequently, since dH is positive on $(0, \bar{x})$, if x' is a common point of increase of F_a and F_b, then it must be that $F_a(x') = F_b(x')$. Now suppose in group 1 player a uses a strategy with support $[0, \bar{x}]$ and player b uses a strategy with support $\{0\} \cup [b, \bar{x}]$, for some $b \in (0, \bar{x})$. Because $\bar{x}$ is in the support of each strategy, both players must obtain the same payoff. On the interval $[b, \bar{x}]$, players' strategies must agree. But then player a's cdf strictly first-order stochastically dominates player b's, from which we obtain the contradiction that, for an effort of zero, player b's expected payoff strictly exceeds player a's.

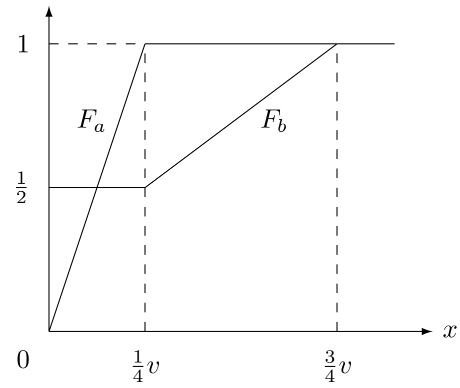

Figure 1: Equilibrium cumulative distribution functions in Example 1.

Next consider player a on team 1. Her payoff from $x \in (0, x_m]$ is $V_a(x) = -x + v\left(1 - \int_x^{\bar{x}} F_b(y)\, dH(y)\right)$. Indifference over $(0, x_m]$ implies $0 = V_a'(x) = -1 + vF_b(x_m)F_b(x_m)\, dF_a(x)$, $\forall x \in (0, x_m)$, from which we obtain, after integration,

$$F_a(x) = \frac{x}{v(F_b(x_m))^2} \qquad \forall x \in [0, x_m]. \tag{9}$$

From (7) and (9) we find $v(F_b(x_m))^2 = x_m = \frac{v}{2}F_b(x_m)$, which in turn implies

$$F_b(x_m) = \frac{1}{2} \qquad \text{and} \qquad x_m = \frac{v}{4}. \tag{10}$$

Using (10) with (8), we find from $F_b(\bar{x}) = 1$ that $\bar{x} = 3v/4$.

From the above analysis, we propose the following as equilibrium strategies:

$$F_a(x) = \begin{cases} 1 & \text{if } x > v/4 \\ \dfrac{4x}{v} & \text{if } x \leq v/4 \end{cases} \qquad \text{and} \qquad F_b(x) = \begin{cases} 1 & \text{if } x > 3v/4 \\ \dfrac{v+4x}{4v} & \text{if } v/4 < x \leq 3v/4 \\ \dfrac{1}{2} & \text{if } x \leq v/4. \end{cases} \tag{11}$$

Construction of these strategies shows that players' payoffs are constant over their strategies' supports. At these strategies, a type-a player earns payoff $u_a^* = 3v/8$ and a type-b player earns payoff $u_b^* = v/4$.

It only remains to show that no player has a profitable deviation. For player a on team 1 contemplating $x \in (v/4, 3v/4]$ the payoff is

$$V_a(x) = -x + v\left(1 - \int_x^{\bar{x}} F_b(y)\, dH(y)\right)$$

$$= -x + v\left(1 - \int_x^{3v/4} \frac{v+4y}{4v} \times \frac{1}{v}\, dy\right)$$

$$= \frac{3v}{8} - \frac{(4x-v)(5v-4x)}{32v}$$

$$< \frac{3v}{8} \qquad \forall x \in (v/4, 3v/4],$$

showing the player a has no profitable deviation. A similar calculation shows player b has no profitable deviation. (We omit the simple verification that inactive players would not choose to become active.) Thus, the foregoing analysis establishes an equilibrium between two teams where each team has one player using F_a and one player using F_b, as given in (11). □

To further our understanding of the extent and effects of free-riding in our setup, we refine the structure of the previous example to obtain, in addition to the number of players, a second source of equilibrium variation. Naturally, agents desire to avoid wasting contribution to a higher effort by a teammate. Intuitively, the probability that an agent wastes his contribution appears large in the semi-symmetric equilibrium in (2), because agents' contributions often overlap. In contrast, (11) describes a situation in which teammates tend to coordinate on different efforts, and this apparently reduces waste. The following Example introduces an intermediate situation.

Example 2 (Equilibrium with interlacing interval supports)**.**

As in Example 1 we assume there are two teams, each with two players, player a and player b. The following strategies are an equilibrium.

$$F_a(x) = \begin{cases} \frac{8}{27} + \frac{x}{v} & \text{if } \frac{10}{27}v \le x \le \frac{19}{27}v \\ \frac{2}{3} & \text{if } \frac{2}{27}v \le x \le \frac{10}{27}v \\ \frac{9x}{v} & \text{if } 0 \le x \le \frac{2}{27}v \end{cases} \quad \text{and} \quad F_b(x) = \begin{cases} 1 & \text{if } \frac{10}{27}v \le x \le \frac{19}{27}v \\ \frac{1}{6} + \frac{9x}{4v} & \text{if } \frac{2}{27}v \le x \le \frac{10}{27}v \\ \frac{1}{3} & \text{if } 0 \le x \le \frac{2}{27}v. \end{cases} \tag{12}$$

While equilibrium proof follows from the general case analyzed in the Appendix, here we report the equilibrium cdfs, depicted in Figure 2. □

Taken together, Proposition 1 and Examples 1 and 2 reveal a second important contrast with the individualistic all-pay auction where all players have common value. Baye *et al.* (1996) found the sum of expected efforts to be constant across all equilibria of the symmetric all-pay contest. By contrast, our examples show that total expected efforts may vary. Even with symmetric teams acting symmetrically, and holding fixed the number of active players, a team's total expected payoff varies over the three equilibria considered, the semi-symmetric equilibrium (see (2) for $N = m = 2$) and the equilibria of Examples 1 and 2. A team's total

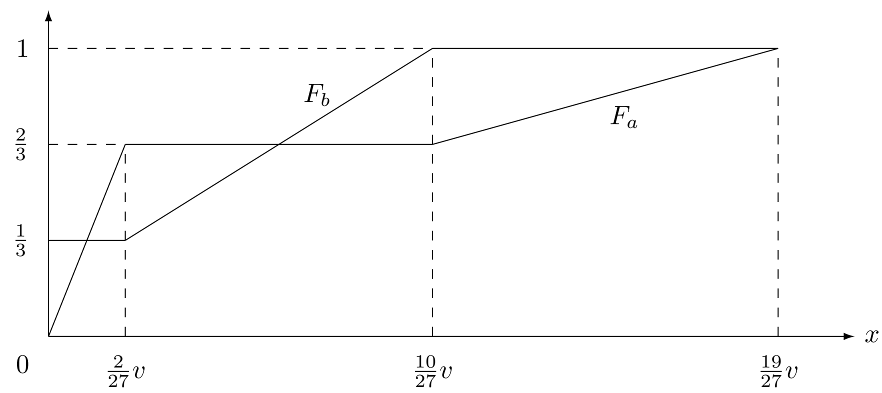

Figure 2: Equilibrium cumulative distribution functions in Example 2.

expected payoff is $135v/216$ in Example 1, $140v/216$ in Example 2, and $144v/216$ in the semi-symmetric equilibrium. Correspondingly, total expected efforts vary across the three equilibria. It is instructive to calculate the expected cost of "wasted contributions," i.e., those contributions overtaken by the other group member's contribution. If one thus defines waste, W, we have

$$W = \int\int \big(x + y - \max\{x, y\}\big)\, dF_a(x)\, dF_b(y)$$
$$= \int x\,(1 - F_b\,(x))\, dF_a\,(x) + \int x\,(1 - F_a\,(x))\, dF_b\,(x)\,, \tag{13}$$

and waste is $v/16 = 0.0625v$ in Example 1, $16v/243 \approx 0.0658v$ in Example 2, and $v/15 \approx 0.0667v$ in the semi-symmetric equilibrium. Surprisingly, while the greater symmetry of strategies within a team is associated with greater waste, it is also associated with higher payoffs as overall efforts are lower.

In the Appendix we generalize Example 2 by presenting equilibria with strategies taking values over an arbitrary number of interlaced intervals. Thus, as we increase the number of intervals, we cover the gamut from the highly asymmetric (within each group) equilibrium in Example 1 to the semi-symmetric equilibrium described by (2). The results presented in the Appendix verify the following intuitive explanation for the existence and the direction of the differences in teams' payoffs outlined above. The finer the interlacing of contribution strategies, the more likely it is that one's contribution is wasted. This decreases an agent's incentive to contribute. However, if all groups coordinate on such an "inefficiently low" level of effort, as the one in the semi-symmetric equilibrium described by (2), then all groups end up benefiting, despite the increased waste, because of the reduced competition across groups. And the strategy profile is not vulnerable to unilateral increases in effort because the within group free-riding prevents any one agent from fully capturing the benefits of additional effort. Therefore, the larger the asymmetry in contributor's equilibrium strategies within each group (that is to say, the coarser the interlacing), the weaker the incentive to free-ride within each group, the *smaller* the payoff for every group, as exemplified above.

4 Asymmetric active groups

4.1 Two-group contests

We now investigate the effects of asymmetry among active groups. We allow active groups to differ in their sizes and values of winning. We return to the original formulation allowing values v_1 and v_2 to differ between groups.[15] We look for an equilibrium where the m active members of group 1 use the same strategy and the n active members of group 2 use the same strategy (obviously, $1 \leq m \leq n_1$ and $1 \leq n \leq n_2$). Let F be the equilibrium cdf used by each active member of group 1, and G the equilibrium cdf used by each active member of group 2.

Now, let F^M be the cdf of the maximum of $m-1$ draws from F; let G^M be the cdf of the maximum of n draws from G. Then $F^M(x) = (F(x))^{m-1}$ and $G^M(x) = (G(x))^n$. With player 1_1 on group 1 exerting effort x, we have

$$\Pr(\text{group 1 wins} \mid x_{1_1} = x) = 1 - \int_x^\infty F^M(y)\, dG^M(y). \tag{14}$$

To understand (14), note that group 2 wins only if its best shot y, say, (having cdf G^M) exceeds x, and then only if all other group 1 members have efforts less than y, which happens with probability $F^M(y)$. Thus, the integral in (14) is just the probability that group 2 wins, given that $x_{1_1} = x$. It now follows that player 1_1's payoff is

$$V_1(x) = -x + v_1 \left[1 - \int_x^\infty F^M(y)\, dG^M(y)\right]. \tag{15}$$

Lemma 1. *The equilibrium strategies F and G (i) are continuous and strictly increasing on their common support $[0, \bar{x}]$, for some $\bar{x} \in (0, \min\{v_1, v_2\}]$; (ii) do not both have atoms at 0; and (iii) are continuously differentiable on $(0, \bar{x})$.*

Standard arguments, as used in Section 3, show F and G have common interval support and cannot have atoms above 0; hence, they are continuous on that support. To complete the proof of Lemma 1 it only remains to establish part (iii), and that is done in the Appendix.

Because V_1 is constant over $[0, \bar{x}]$, it is differentiable on $(0, \bar{x})$, with $0 \equiv V_1'(x) = -1 + v_1 F^M(x)\, dG^M(x)$, which can be rewritten as

$$\frac{1}{nv_1} = \Big(F(x)\Big)^{m-1} \Big(G(x)\Big)^{n-1} G'(x), \tag{16}$$

thus providing a differential equation for G. Analogous reasoning for an active member of group 2 yields a

[15] While Baik *et al.* (2001) and Chowdhury *et al.* (2013a) allow values to vary between and within groups, they focus on equilibria in which only one player on a group is active, so the values of inactive players are not relevant.

second differential equation:

$$\frac{1}{mv_2} = \Big(F(x)\Big)^{m-1}\Big(G(x)\Big)^{n-1}F'(x). \tag{17}$$

From (16) and (17) we conclude that

$$G'(x) = \left(\frac{v_2/n}{v_1/m}\right)F'(x), \tag{18}$$

for $x > 0$. Let $\bar{x}$ denote the (common) upper end of the support of the cdfs. We now have

$$1 - G(x) = \int_x^{\bar{x}} G'(y)\,dy = \left(\frac{v_2/n}{v_1/m}\right)\int_x^{\bar{x}} F'(y)\,dy = \left(\frac{v_2/n}{v_1/m}\right)(1 - F(x)),$$

or

$$G(x) = \left(1 - \frac{v_2/n}{v_1/m}\right) + \left(\frac{v_2/n}{v_1/m}\right)F(x). \tag{19}$$

In particular, if $v_1/m > v_2/n$, then G puts an atom on 0 while F does not; moreover, the strategy of an active player in group 1 strictly first-order stochastically dominates that of an active player in group 2. Thus, active group 1 players exert greater effort than active group 2 players when $\frac{v_1}{m} > \frac{v_2}{n}$, that is, when, for example, group 1 has a relatively larger values or relatively less diffuse responsibility for effort.

Returning to the final determination of F and G, with $mv_2 \leq nv_1$ we proceed as follows. Substitute G from (19) into (17) to obtain the following differential equation for F:

$$\frac{1}{mv_2} = \Big(F(x)\Big)^{m-1}\left[\left(1 - \frac{v_2/n}{v_1/m}\right) + \left(\frac{v_2/n}{v_1/m}\right)F(x)\right]^{n-1}F'(x). \tag{20}$$

Solve this for F, with the boundary condition $F(0) = 0$. Then determine $\bar{x}$ as the solution to $F(x) = 1$, and, finally, determine G using (19). By construction of the cdfs F and G, active players have no profitable deviations. To conclude the verification that the proposed strategies constitute an equilibrium, we must check that no inactive player has a profitable deviation. The following lemma, proven in the Appendix, does so. Moreover, it shows that a unique pair of functions F and G is identified by the previous procedure.

Lemma 2 (Equilibrium and unique solution)**.** *Suppose $v_1/m \geq v_2/n$. There is a unique solution F to the differential equation* (20) *with initial condition $F(0) = 0$. Therefore,* (19) *identifies a unique G. Moreover, F and G determined through* (19) *and* (20) *describe an equilibrium.*

It is natural to ask which group is more likely to win. Giving a general answer turns out to be nontrivial. We proceed in incremental steps by first analyzing how the equilibrium strategies identified in Lemma 2 vary with values, v_1 and v_2, and the numbers of active players, m and n. We provide a partial analysis by rewriting (20) as

$$1 = v_2 \left[1 - \frac{v_2/n}{v_1/m}\left(1 - \left[H(x)\right]^{1/m}\right)\right]^{n-1} H'(x), \tag{21}$$

where $H(x) \equiv (F(x))^m$ denotes the cdf of group 1's best-shot cdf and H satisfies the boundary condition $H(0) = 0$. The differential equation (21) yields some comparative statics results for group-1's best-shot cdf.

Proposition 3. *Suppose $v_1/m > v_2/n$. Then decreasing m or increasing v_1 causes the distribution of group 1's best shot derived earlier to shift to the right in the sense of first-order stochastic dominance. Correspondingly, the individual efforts of active group-1 players are also larger in the sense of first-order stochastic dominance.*

With (19) and (21) we can now calculate the groups' probabilities of winning. Given $v_1/m > v_2/n$, we have the cdf of group 1's best shot given by $(F(\cdot))^m$ and group 2's by $(G(\cdot))^n$, so

$$\begin{aligned}
\Pr(\text{group 1 wins}) &= \int_0^{\bar{x}} [G(y)]^n \frac{d}{dy}\left([F(y)]^m\right) dy \\
&= \int_0^{\bar{x}} \left[\left(1 - \frac{v_2/n}{v_1/m}\right) + \left(\frac{v_2/n}{v_1/m}\right) F(y)\right]^n \frac{d}{dy}\left([F(y)]^m\right) dy \\
&= \int_0^1 \left[1 - \left(\frac{mv_2}{nv_1}\right)\left(1 - z^{1/m}\right)\right]^n dz.
\end{aligned} \tag{22}$$

The integrand in (22) is clearly increasing in v_1 and decreasing in v_2. As shown in the proof of Proposition 3 we also see the integrand is decreasing in m and, as shown in the proof of the following proposition, increasing in n. Therefore, we have the following.

Proposition 4. *Suppose $v_1/m > v_2/n$. At the semi-symmetric equilibrium earlier derived, group 1's probability of winning increases as v_1 increases, n increases, v_2 decreases, or m decreases.*

The results for v_1 and v_2 are intuitively reasonable: increasing v_1 makes winning more desirable for active group-1 players, and reducing m lessens the incentives to free ride. Both effects increase the probability that group 1 wins. Similarly, reducing v_2 or increasing n weakens group 2 and makes it less likely to win. And while the integral in (22) has no convenient closed form, for any particular values of m and n it is easily calculated. Table 1 provides examples of such calculations, reflecting the conclusions of Proposition 4. Also, as illustrated in the table, it can be shown more generally that, for fixed v_1, v_2, and m, as n becomes arbitrarily large, group 1's probability of winning is bounded away from 1. This occurs because the distribution of group 2's best-shot effort approaches a nondegenerate distribution.

One situation in which the groups' chances of winning are clearly determined is when groups are very large. Suppose both groups grow large, say in constant proportion: $m = tn$, where t satisfies $1 \geq mv_2/nv_1 = tv_2/v_1$,

Table 1: —— Insert: The probability that group 1 wins ——

that is, $t \le v_1/v_2$. Then we have by the Bounded Convergence Theorem that

$$
\begin{aligned}
\lim_{n\to\infty} \Pr(\text{group 1 wins} \,|\, m = tn) &= \int_0^1 \lim_{n\to\infty} \left[\left(1 - \frac{tv_2}{v_1}\right) + \left(\frac{tv_2}{v_1}\right) z^{1/(tn)} \right]^n dz \\
&= \int_0^1 z^{v_2/v_1}\, dz \\
&= \frac{v_1}{v_1 + v_2}. \qquad (23)
\end{aligned}
$$

Remarkably, (23) is independent of t; so as the two groups' active groups become arbitrarily large, the probabilities of winning become independent of the relative sizes of the active groups, only relative values matter.[16]

While the equilibrium identified in Lemma 2 above using (19) and (20) does not yield convenient closed-form solutions for arbitrary m and n, v_1 and v_2, we consider two examples where it does. The next proposition derives explicit solutions where neither group places an atom on 0 and all active players use the same strategy.

Proposition 5. *Suppose $v_1/m = v_2/n$. Then it is an equilibrium for all active players to use the same strategy F, which is given by*

$$
F(x) = \left[\left(\frac{m+n-1}{mv_2} \right) x \right]^{\frac{1}{m+n-1}} \quad \text{on} \quad \left[0, \frac{mv_2}{m+n-1}\right] \quad \left(= \left[0, \frac{nv_1}{m+n-1}\right] \right).
$$

Group 1 wins with probability $\frac{v_1}{v_1+v_2}$, group 2 with probability $\frac{v_2}{v_1+v_2}$. The expected payoff is $\left(\frac{m-1}{m+n-1}\right) v_1$ to each active group 1 member, $\left(\frac{n-1}{m+n-1}\right) v_2$ to each active group 2 member.

Beyond determining a ranking of expected probability of winning, since equilibrium is in mixed strategies it is interesting to establish a stronger, first-order stochastic dominance comparison. When $v_1/m > v_2/n$ it is immediate from (19) that the distribution of an active group-1 player first-order stochastically dominates that of an active group-2 player. However, this does not yet establish whether the best shot of group 1 first-order stochastically dominates that of group 2. Given the conditions of Proposition 5, all active players use the same strategy. When $v_1/m = v_2/n$, we see $v_1/v_2 = m/n$, so individuals in the larger active group also value the prize more highly. Therefore, the cdf of the best shot for the group valuing winning more highly strictly first-order stochastically dominates the cdf for the best shot of the other group. The following

[16] The limit in (23) is given with the active groups keeping their relative sizes constant. The same limit probability obtains taking separately the limits as $n \to \infty$ and then $m \to \infty$.

proposition extends this stochastic-dominance comparison between groups' best shots to the case where $v_1/m > v_2/n$.[17]

Proposition 6. *Suppose* $(m,n) \in \{(i,j) \,|\, 1 \le i \le n_1, 1 \le j \le n_2\}$ *and suppose* $v_1/m > v_2/n$. *Let* F *and* G *be determined through Lemma 2. Then there exists an equilibrium in which group 1 has* m *active members using cdf* F, *group 2 has* n *active members using cdf* G, *and all other players are inactive. Let* F^{max} *and* G^{max} *denote the cdfs of the maximum efforts of groups 1 and 2, respectively.*
(i) If $v_1 \ge v_2$, *then* F^{max} *strictly dominates* G^{max} *in the sense of first-order stochastic dominance.*
(ii) If $v_1 < v_2$, *then* F^{max} *and* G^{max} *cannot be ranked by first-order stochastic dominance.*

It is interesting to note that, even when best shots cannot be ranked by first-order stochastic dominance, it is the case that the group valuing winning more highly is expected to exert greater total effort. This follows from (18), which implies

$$\underbrace{n \times \mathrm{E_G}(x)}_{\text{total expected group-2 efforts}} = \frac{v_2}{v_1} \times \underbrace{m \times \mathrm{E_F}(x).}_{\text{total expected group-1 efforts}}$$

Thus, group 2 exerts larger total expected effort than group 1 if and only if $v_2 > v_1$. Indeed, the ratio of total (expected) group expenditures equals the ratio of values.

Beyond Proposition 5, a second case in which equilibrium strategies can be exactly derived is where one player faces many. The example again highlights the importance of v_1/m versus v_2/n.

Example 3 (One versus many: $m = 1$)**.**

Part i): $v_1 > v_2/n$. By (19), active group-2 players will place an atom on 0. With $m = 1$, (16) becomes

$$\frac{1}{nv_1} = \Big(G(x)\Big)^{n-1} G'(x) = \frac{d}{dx}\Big(G(x)\Big)^n;$$

integrate the previous equation to obtain

$$\frac{x}{v_1} = \Big(G(x)\Big)^n - \Big(G(0)\Big)^n = \Big(G(x)\Big)^n - \left(1 - \frac{v_2}{nv_1}\right)^n,$$

where the second equality uses (19), with $F(0) = 0$. Therefore, $G(x) = \left[\frac{x}{v_1} + \left(1 - \frac{v_2}{nv_1}\right)^n\right]^{\frac{1}{n}}$. Solving $G(\bar{x}) = 1$ we find

$$\bar{x} = \left(1 - \left(1 - \frac{v_2}{nv_1}\right)^n\right) v_1, \tag{24}$$

[17] Propositions 5 and 6 cover all the possibilities. When $v_1/m < v_2/n$, simply apply Proposition 6 with the roles of groups 1 and 2 interchanged.

which is strictly less than v_1 (because $v_2 < nv_1$) and, if $n \geq 2$, is less than v_2.[18] Consequently, when $n \geq 2$ the one active player on group 1 actually achieves a strictly positive expected payoff, *even if* $v_1 < v_2$—the free riding among active players on group 2 opens the door to this possibility. Indeed, as n increases, each active group-2 player is increasingly likely to exert 0 effort; moreover, the equilibrium G becomes larger, corresponding to stochastically lower equilibrium individual efforts. The equilibrium F is found from (19) to be

$$F(x) = \frac{nv_1}{v_2}\left(G(x) - \left(1 - \frac{v_2}{nv_1}\right)\right) = \frac{nv_1}{v_2}\left(\left[\left(1 - \frac{v_2}{nv_1}\right)^n + \frac{x}{v_1}\right]^{1/n} - \left(1 - \frac{v_2}{nv_1}\right)\right).$$

Next, using (22) we explicitly calculate the probability that group 1 wins as

$$\Pr(\text{group 1 wins}) = \int_0^1 \left[\left(1 - \frac{v_2}{nv_1}\right) + \frac{v_2}{nv_1}z\right]^n dz = \frac{n}{n+1} \times \frac{v_1}{v_2}\left(1 - \left(1 - \frac{v_2}{nv_1}\right)^{n+1}\right).$$

Here in part i, v_1/v_2 ranges from $1/n$ to ∞. By Proposition 4 therefore, we see that group 1's probability of winning increases with its relative value (v_1/v_2) in the contest of one active player against many, and this probability can be anywhere from $1/(n+1)$ to 1. In contrast, if $v_1 = v_2$, then one can show that group 1's probability of winning increases in n, but even as the number of active group-2 players grows arbitrarily large, group 1's probability of winning remains bounded above by $1 - e^{-1} \approx 0.6321$, well below 1.

Part ii): $v_1 < v_2/n$. Rearranging (19) we obtain

$$F(x) = \left(1 - \frac{nv_1}{v_2}\right) + \frac{nv_1}{v_2}G(x). \tag{26}$$

Now we see player 1 places an atom on 0; hence, player 1 will earn payoff 0 in equilibrium. Therefore,

$$0 = -x + v_1(G(x))^n,$$

which implies $G(x) = (x/v_1)^{1/n}$ and $\bar{x} = v_1$ for the active group-2 players. And now the active group-1 player uses strategy

$$F(x) = \left(1 - \frac{nv_1}{v_2}\right) + \left(\frac{x}{v_1}\right)^{1/n}$$

on $[0, v_1]$. Here in part ii, group 1's value is very low relative to group 2's, even accounting for the attendant free riding among active group-2 players. The best the active player in group 1 can do is to earn a zero payoff.[19] □

[18] For $n = 1$, we have the standard asymmetric all-pay auction and $\bar{x} = v_2$. From (24) we see $\bar{x} < v_2$ if and only if

$$1 - \frac{v_2}{v_1} < \left(1 - \frac{v_2}{nv_1}\right)^n. \tag{25}$$

The right-hand side of (25) is strictly increasing in n. Since the left-hand and right-hand sides of (25) are equal when $n = 1$, it follows that the strict inequality holds for $n \geq 2$.

[19] The case of $v_2 = nv_1$, covered by Proposition 5, also leaves the active group 1 player with a zero payoff.

4.2 Three-group contests

We conclude with an example involving three groups, not all identical, to illustrate the consequences of having more than one active member per group on the incentives of low-value groups to participate.

Example 4 (An example with three active groups, $v_1 = v_2 = v \geq v_3$).

Suppose groups 1 and 2 have 2 active members, and group 3 has none. From Proposition 1 we see that the active players would randomize over $[0, 2v/3]$. Therefore, if $v_3 < 2v/3$, group 3 players are effectively "blockaded" from active participation.[20] But what happens if $v_3 > 2v/3$? It turns out that two cases are to be considered, $v_3 \in [2v/3, 3v/4)$ and $v_3 \in [3v/4, v]$.

Part i): $v_3 \in [3v/4, v]$. So now suppose groups 1 and 2 have 2 active members, group 3 has 1. We look for an equilibrium that is symmetric across groups 1 and 2, with agents in these groups exerting effort using cdf $F_1 = F_2 = F$, while the active agent in group 3 uses F_3. In the Appendix, we show the following is an equilibrium:

$$F(x) = \left(\frac{v_3 - \bar{x} + x}{v_3}\right)^{1/4} \quad \forall x \in [0, \bar{x}], \quad \text{and}$$
$$F_3(x) = \frac{4(v_3)^{3/4}}{3v}(v_3 - \bar{x} + x)^{1/4}\left[1 - \left(\frac{v_3 - \bar{x}}{v_3 - \bar{x} + x}\right)^{3/4}\right] \quad \forall x \in [0, \bar{x}],$$

where

$$\bar{x} = v_3 - \frac{2^{1/3}}{8v_3^{1/3}}\left(4v_3 - 3v\right)^{4/3}.$$

If $v_3 = v$, one can show that F_3 stochastically dominates $(F)^2$. Therefore, group 3 has a larger probability of winning than does either group 1 or group 2. This reinforces the earlier finding in the two-group contest that when values are equal, the smaller active group is more likely to win. And, if one takes $v_3 < v$ sufficiently close to v, continuity implies that the low-value group 3 remains the most likely to win.

Part ii): $v_3 \in [2v/3, 3v/4)$. From part i, one sees that player 3's payoff is zero when $v_3 = 3v/4$. Hence, for smaller v_3 player 3's expected payoff will also be zero. Therefore, for $v_3 \in (2v/3, 3v/4)$ player 3 will choose to participate in the contest but only earns a payoff of zero. Correspondingly, equilibrium will require $\bar{x} = v_3$. Surprisingly, it turns out that in equilibrium player 3 will not randomize over the entire interval $[0, v_3]$. Rather, groups 1 and 2 randomize over $[0, v_3]$, but group 3 continuously randomizes only over $[b, v_3]$

[20] If groups 1 and 2 just use their semi-symmetric strategy from (2), then $(F(x))^4$ is strictly convex, with support $[0, 2v/3]$. Therefore, if $v_3 < 2v/3$, then over $[0, v_3]$ the group-3 player would want to use either $x = 0$ or $x = v_3$; the former gives a zero payoff, the latter a strictly negative payoff. So for $v_3 < 2v/3$, group-3 players would choose not to participate even when groups 1 and 2 ignore group 3.

and puts all remaining mass on 0.[21]

In the Appendix, we show the following is an equilibrium:

$$F(x) = \begin{cases} \left(\dfrac{x}{v_3}\right)^{1/4} & x \in [b, v_3] \\ \dfrac{x^{1/3}}{v_3^{1/4} b^{1/12}} & x \in [0, b) \end{cases}$$

and

$$F_3(x) = \begin{cases} \sqrt{\dfrac{v_3}{x}} \left(1 - \dfrac{4v_3}{3v} + \dfrac{4v_3^{1/4}}{3v} x^{3/4}\right) & x \in [b, v_3] \\ \dfrac{3}{v}\left(\dfrac{v_3}{2}\right)^{2/3} (3v - 4v_3)^{1/3} & x \in [0, b), \end{cases}$$

where b is given by

$$b = \left[\frac{6v}{(v_3)^{1/4}}\left(1 - \frac{4v_3}{3v}\right)\right]^{4/3}.$$

Consequently b takes value 0 when $v_3 = 3v/4$ (consistent with part i, where player 3 uses no atoms) and equals $2v/3$ when $v_3 = 2v/3$; in the limit as v_3 approaches $2v/3$ from above, group three effectively drops out—$F(b)$ goes to 1 as v_3 goes to $2v/3$. □

Example 4 reveals a final important contrast with the individualistic all-pay auction contest. There, with players having values $v_1 = v_2 > v_3$, Baye *et al.* (1996) show the unique equilibrium has players 1 and 2 earn zero payoffs and player 3 remain inactive. Example 4 shows neither of these conclusions necessarily applies to group contests. Because groups 1 and 2 have more than one active player, their players earn positive payoffs, which means the maximum effort of players in groups 1 and 2 is less than v_1. This raises the possibility that if v_3 is not too much smaller than v_1, then a player from group 3 could actively participate, even earning a positive payoff, as in part (i) of Example 4.

5 Conclusion

We have analyzed a group contest problem in which individual members' efforts are aggregated into a group effort via the best-shot technology. The interplay of within-group free-riding and across-group competition turns out to be especially interesting in this framework because it makes possible the existence of a wide variety of equilibria.

[21]This strategy form is analogous to those in the "additional equilibria" uncovered by Baye *et al.* (1996). Unlike the equilibria of Baye *et al.*, however, here b is not a free parameter but is (uniquely) determined as part of the equilibrium.

Indeed, the group contest acquires almost a coordination game flavor: equilibria with large group efforts coexist along with equilibria with small group efforts (and with many intermediate cases). The best-shot technology is key to this result, along with the fact that the prize is a public good within a group. Indeed, as known from the theory of best-shot public goods (*e.g.*, Barbieri and Malueg, 2012), the largest group effort occurs when only one agent within a group is active, while the smallest group effort arises when all agents within the group are active. In the former case, the lone active agent in a group does not find it profitable to lower his effort from this high (average) level because his group's effort would be lowered one-to-one, and so he would bear the whole impact of the decreased probability of winning. In the latter case, one of the many active agents within a group does not find it profitable to increase his effort from his low (average) level because the probability of his being the group's best-shot is small, since all other members of the same group are contributing as well; thus, the marginal positive effect on the probability of his group's winning is small.

This wide variety of equilibria makes our results qualitatively different from those of the individualistic all-pay auction of Baye *et al.* (1996): we find rents are not necessarily dissipated in equilibrium, total expected efforts vary across equilibria, and participation is expected to be larger. An intriguing finding is that, in contrast to many results in the literature, free-riding can be beneficial for players as it reduces competition among groups. Less free-riding, in contrast, results in a cut-throat competition where the total value of the prize is the same but the total amount of effort expended is higher. Some equilibria derived here had never been, to the best of our knowledge, documented in the contest literature. In particular, equilibrium strategies with interlacing supports are entirely new, having been identified in neither group nor individualistic contests. We use these equilibria to identify another source of free-riding within groups.

Many aspects of best-shot group contest problem deserve further attention. For instance, we have explored neither the consequences of preference asymmetries within groups nor of the possibility of sequential contributions within a group.[22] Both features are likely to be relevant for applied analysis, for instance in determining the effects of the different state-by-state rules concerning *amicus curiae* briefs. In this paper we have also abstracted from private information. These issues are the subjects of current research.

[22]It is however easy to show that equilibria with multiple active agents per group exist even when values within a group are not exactly identical, for instance in situations similar to that in Example 1. Therefore, the source of the most relevant differences between our results and those of Baik *et al.* (2001) and Chowdhury *et al.* (2013a) is robust to small, within-group asymmetries.

Appendix

Proof of Proposition 1. To complete the discussion in the main body of the paper and prove the proposition, we must show that no inactive players would wish to become active. Because teams are acting symmetrically, each of the inactive players earns payoff v/N. An inactive player considering a deviation would only consider efforts levels in $(0, \bar{x}]$. The payoff to such a deviator would be

$$\begin{aligned} V_D(x) &= -x + v\left[[F(x)]^{Nm} + \frac{m}{Nm}\left(1 - [F(x)]^{Nm}\right)\right] \\ &= -x + v\left[\frac{1}{N} + \frac{N-1}{N}[F(x)]^{Nm}\right]. \end{aligned} \tag{27}$$

Because $[F(x)]^{Nm}$ is proportional to $x^{\frac{Nm}{Nm-1}}$, we see that V_D is strictly convex on $[0, \bar{x}]$, implying that an inactive player's best effort level will be either 0 or $\bar{x}$. And his payoff from effort level $\bar{x}$ is

$$v - \bar{x} = v - \left(\frac{Nm - m}{Nm - 1}\right)v = \frac{m-1}{Nm-1}v < \frac{v}{N}.$$

Consequently, becoming active is unprofitable—an inactive player has no profitable deviation. □

Generalization of Example 2: equilibria with interlacing interval supports. We use the same notation and terminology of Example 2 and we consider an arbitrary odd number k of disjoint intervals covering $(0, \bar{x})$ (a similar construction can be carried out for even values of k). As in Example 1 we assume there are two teams, each with two players, one type-a and one type-b. The support of a's equilibrium strategy is $[0, x_1] \cup [x_2, x_3] \cup ... \cup [x_{k-3}, x_{k-2}] \cup [x_{k-1}, \bar{x}]$, while the support of b's equilibrium strategy is $\{0\} \cup [x_1, x_2] \cup [x_3, x_4] \cup ... \cup [x_{k-2}, x_{k-1}]$.

We first determine the values of these interval extremes, along with equilibrium strategies F_a, F_b and the groups' best shot cdf $H = F_a \cdot F_b$. Then we perform a comparative statics exercise with respect to k that formally demonstrates how, as we decrease k, the asymmetry in contributor's equilibrium strategies within each group increases, the incentive to free-ride within each group diminishes, and that the payoff for every group becomes *smaller*.

We begin with the determination of F_a and F_b, and thereby H. We start with the FOC for type-a's utility maximization relative to a point x belonging to one of the intervals in which F_a is strictly increasing and $F_b > 0$, say $[x_{k'-1}, x_{k'}]$ (in other words, take k' odd and $k' < k-1$). This is nothing but equation (5), which in our context yields $1 = vF_b(x_{k'})h(x)$, where we denote $h(x) = dH(x)$. Therefore, by continuity of H, integration yields

$$H(x_{k'}) - H(x_{k'-1}) = \frac{x_{k'} - x_{k'-1}}{vF_b(x_{k'})}. \tag{28}$$

At the same time, b must be indifferent between contributing $x_{k'}$ and $x_{k'-1}$, leading to

$$\begin{aligned}
x_{k'} - x_{k'-1} &= v \int_{x_{k'-1}}^{x_{k'}} F_a(y)\, dH(y) \\
&= v F_b(x_{k'-1}) \int_{x_{k'-1}}^{x_{k'}} F_a(y)\, dF_a(y) \\
&= \frac{v}{2} F_b(x_{k'-1}) \left[(F_a(x_{k'}))^2 - (F_a(x_{k'-1}))^2 \right] \\
&= \frac{1}{2} \frac{v}{F_b(x_{k'-1})} \left[H^2(x_{k'}) - H^2(x_{k'-1}) \right], \qquad (29)
\end{aligned}$$

where the last equality uses $F_b(x_{k'}) = F_b(x_{k'-1})$. Combining (28) and (29), and again noting that $F_b(x_{k'-1}) = F_b(x_{k'})$, we derive

$$2F_b(x_{k'-1}) = F_a(x_{k'}) + F_a(x_{k'-1}). \qquad (30)$$

Similarly, when considering intervals $[x_{k'-2}, x_{k'-1}]$ in which F_b is strictly increasing, one obtains

$$2F_a(x_{k'-2}) = F_b(x_{k'-1}) + F_b(x_{k'-2}). \qquad (31)$$

There are $(k-1)/2$ equalities described by (30) and $(k-1)/2 - 1$ equalities described by (31). Moreover, we have the two boundary conditions $F_b(x_{k-1}) = 1$, and $F_a(0) = 0$. Finally, note that, in addition to $F_a(0)$, only $\frac{k-1}{2}$ other values of F_a need to be determined, since $F_a(x_{k'+1}) = F_a(x_{k'})$ for an odd k', and that similar considerations hold for b. Therefore, maintaining k' odd and $k' < k-1$, the solution of this linear system of equations is

$$F_a(x_{k'+1}) = \frac{k'+1}{k} = F_a(x_{k'}), \qquad (32)$$

and

$$F_b(x_{k'}) = \frac{k'}{k} = F_b(x_{k'-1}). \qquad (33)$$

Indeed, to verify (30) is satisfied, note that,

$$2F_b(x_{k'-1}) = 2\left(\frac{k'}{k}\right) = \frac{k'+1}{k} + \frac{k'-1}{k} = F_a(x_{k'}) + F_a(x_{k'-1}),$$

while to verify (31) we see that

$$2F_a(x_{k'-2}) = 2\left(\frac{k'-1}{k}\right) = \frac{k'}{k} + \frac{k'-2}{k} = F_b(x_{k'-1}) + F_b(x_{k'-2}).$$

Conditions (32) and (33), when used together with (5), further imply that

$$f_a(x) = \frac{1}{v}\left(\frac{k}{k'}\right)^2 \text{ for } x \in [x_{k'-1}, x_{k'}],\ f_b(x) = \frac{1}{v}\left(\frac{k}{k'-1}\right)^2 \text{ for } x \in [x_{k'-2}, x_{k'-1}],$$

and that, for all $\tilde{k} \leq k-1$, we have

$$H(x_{\tilde{k}}) = \frac{\tilde{k}(\tilde{k}+1)}{k^2}. \tag{34}$$

The above ensures that a is indifferent between contributing any amount $x \in [0, x_1] \cup [x_2, x_3] \cup ... \cup [x_{k-3}, x_{k-2}] \cup [x_{k-1}, \bar{x}]$. To see that a does not benefit from a contribution that b also makes, note that for any x in an interval in which F_b is strictly increasing, the first derivative of a's utility, using (5), is

$$-1 + vF_b(x)h(x),$$

and $h(x)$ is constant, since both $f_b(x)$ and $F_a(x)$ are constant in this interval. Therefore, a's utility is strictly convex in an interval in which F_b is strictly increasing, so its maximum is at the extremes of the interval. But the above reasoning already ensures that a does not strictly prefer any contribution belonging to $\{x_1, x_2, ..., x_{k-1}, \bar{x}\}$ to the equilibrium strategy. Similar considerations apply to b.

We now proceed to determining the values of $x_1, x_2, ..., x_{k-1}, \bar{x}$. By (28) and (34), we obtain that, for k' odd,

$$v \cdot \frac{2\,(k')^2}{k^3} = x_{k'} - x_{k'-1}.$$

For $\tilde{k}$ even, the necessary and sufficient condition to maximize type b's utility leads to the following analogue of (28):

$$vF_a\left(x_{\tilde{k}}\right)\left(H(x_{\tilde{k}}) - H(x_{\tilde{k}-1})\right) = x_{\tilde{k}} - x_{\tilde{k}-1}.$$

Equation (34) then yields

$$v\frac{\tilde{k}}{k}\frac{2\tilde{k}}{k^2} = x_{\tilde{k}} - x_{\tilde{k}-1};$$

therefore, the size of each interval, except for $\bar{x} - x_{k-1}$, follows the same formula (but intervals are all of different length). Starting from $x_0 = 0$, then, it is therefore possible to determine all intervals endpoints up to x_{k-1}. Finally, $\bar{x}$ is determined from the indifference condition of type-a when contributing $x \in [x_{k-1}, \bar{x}]$, i.e., $vH(x) - x = v - \bar{x}$, which at $x = x_{k-1}$ implies

$$\bar{x} = x_{k-1} + v\left(1 - H\left(x_{k-1}\right)\right) = x_{k-1} + v\left(1 - \frac{(k-1)\,k}{k^2}\right) = x_{k-1} + \frac{v}{k}.$$

Because the support intervals have union $[0, \bar{x}]$ and they are disjoint except possibly at endpoints, it follows that the sum of their lengths is $\bar{x}$; therefore,

$$\bar{x} = v\frac{2}{k^3}\sum_{j=1}^{j=k-1}(j)^2 + \frac{v}{k} = \frac{2v}{k^3}\frac{(k-1)(k)(2k-1)}{6} + \frac{v}{k} = v\frac{2k^2+1}{3k^2}.$$

We now perform a comparative statics exercise by varying the number of intervals, i.e., k. Note that, as a function of k, $\bar{x}$ is decreasing and that it converges to $\frac{2}{3}v$ as k goes to infinity. Moreover, the payoff of type a is $v - \bar{x} = v(k^2-1)/(3k^2)$, while the payoff of b is

$$\begin{aligned}
v\left(1 - \int_{x_{k-1}}^{\bar{x}} F_a(y)\,dH(y)\right) - x_{k-1} &= v\left(1 - \frac{1}{2}\left(1 - \left(H(x_{k-1})\right)^2\right)\right) - x_{k-1} \\
&= v\left(1 - \frac{1}{2}\left(1 - \left(\frac{k(k-1)}{k^2}\right)^2\right)\right) - \left(\bar{x} - \frac{v}{k}\right) \\
&= v\left(1 + \frac{1}{2k^2} - \frac{1}{k}\right) - \bar{x} + \frac{v}{k} \\
&= v - \bar{x} + \frac{v}{2k^2} \\
&= v\left(\frac{2k^2+1}{6k^2}\right).
\end{aligned}$$

Therefore, the difference in payoffs within the group is $v/2k^2$ and the total payoff of the group is

$$2(v-\bar{x}) + \frac{v}{2k^2} = v\left(2\frac{k^2-1}{3k^2} + \frac{1}{2k^2}\right) = v\frac{4k^2-1}{6k^2} = v\left(\frac{2}{3} - \frac{1}{6k^2}\right).$$

Clearly, the difference in payoffs within the group is decreasing in k, while the sum is increasing. Also, as $k \to \infty$, the difference in payoffs converges to zero, while the sum converges to $2v/3$, which is the group's payoff in the fully symmetric equilibrium described in equation (2).[23] Also, note that the equilibrium is symmetric across groups, so the expected probability of winning for each group remains 1/2 for any k. Since the expected payoff is increasing in k, the expected total group effort must therefore be decreasing in k (an explicit calculation follows). And since the difference in payoffs within the group is decreasing in k and converges to zero, expected individual efforts converge to the same limit.

As in (13) we measure waste as the expected cost of contributions beaten by the other member of a group:

$$W = \int x(1 - F_b(x))\,dF_a(x) + \int x(1 - F_a(x))\,dF_b(x).$$

[23] Indeed, one can show that the equilibrium strategies converge to the one described in (2). Details are available upon request.

For instance, in Example 2 we have that waste is

$$\int_0^{\frac{2}{27}v} x \cdot \frac{2}{3} \cdot \frac{9}{v}\, dx + \int_{\frac{2}{27}v}^{\frac{10}{27}v} x \cdot \frac{1}{3} \cdot \frac{9}{4v}\, dx = \frac{16}{243}v.$$

To calculate how W depends on k in general, we begin by rewriting W as

$$W = \int x\, d\left(F_a(x) + F_b(x)\right) - \int x\, dH(x).$$

The first term is expected total effort of the group. It can be calculated indirectly from the payoff calculations in the paper. Since a group's expected winnings are v while payoff is calculated above as $v\left(\frac{2}{3} - \frac{1}{6k^2}\right)$, it follows that the expected total effort is $\left(2k^2+1\right)v/6k^2$. The term $\int x\, dH(x)$ is usefully decomposed as $\sum_{j=1}^{k-1} \int_{x_{j-1}}^{x_j} x\, dH(x) + \int_{x_{k-1}}^{\bar{x}} x\, dH(x)$, and exploiting the fact that H is conditionally uniform over (x_{j-1}, x_j), we have

$$\begin{aligned}
\int x\, dH(x) &= \sum_{j=1}^{k-1} \frac{x_j + x_{j-1}}{2}\left(H(x_j) - H(x_{j-1})\right) + \frac{\bar{x} + x_{k-1}}{2}\left(1 - H(x_{k-1})\right) \\
&= \sum_{j=1}^{k-1} \frac{1}{3}\frac{v}{k^3} j\left(2j^2+1\right)\left(2\frac{j}{k^2}\right) + \frac{v}{6k^3}\left(4k^2 - 3k + 2\right) \\
&= \frac{5k^2 + 8k^4 + 2}{30k^4}v,
\end{aligned}$$

where the second equality uses the formulas for x_j and $H(x_j)$ developed earlier. Therefore, waste is calculated as

$$W = \frac{\left(k^2+1\right)(k+1)(k-1)}{15k^4}v.$$

which is easily seen to be a strictly increasing function of k. Also, as $k \to \infty$, $W \to \frac{1}{15}v$, which one can show is the waste in the semi-symmetric equilibrium. □

Proof of Lemma 1. Expression (15) and Lemma 1 show $V_1(x)$ is continuous. If v_1^* denotes the equilibrium payoff to one of the m active players using cdf F, then it must be that $V_1(x) = v_1^*$ on a dense subset of $[0, \bar{x}]$; and because V_1 is continuous, it follows that V is constant on $[0, \bar{x}]$. Hence, V_1 is continuously differentiable on $(0, \bar{x})$. Rearranging the expression for V_1, we obtain

$$1 - \frac{V_1(x) + x}{v_1} = \int_x^{\infty} F^M(y)\, dG^M(y).$$

Since the above left-hand side is continuously differentiable, so will be the right-hand side. And this implies, too, that $G^M \in C'$ for $x \in (0, \bar{x})$. To see this, using the definition of derivative and $V_1' = 0$, we have

$$-\frac{1}{v_1} = \lim_{h\to 0} \frac{\int_{x+h}^{\infty} F^M(y)\, dG^M(y) - \int_x^{\infty} F^M(y)\, dG^M(y)}{h} = -\lim_{h\to 0} \frac{\int_x^{x+h} F^M(y)\, dG^M(y)}{h},$$

and, since F^M is continuous, we obtain[24]

$$\frac{1}{v_1} = \lim_{h\to 0} F^M(x_h') \frac{\int_x^{x+h} dG^M(y)}{h} = \lim_{h\to 0} F^M(x_h') \frac{G^M(x+h) - G^M(x)}{h},$$

where $x_h' \in [x, x+h]$, which, by the usual properties of limits, leads to

$$\frac{1}{v_1} \frac{1}{F^M(x)} = \lim_{h\to 0} \frac{G^M(x+h) - G^M(x)}{h}. \tag{35}$$

The above demonstrates that G^M, and therefore also G, is continuously differentiable on $(0, \bar{x})$. Analogous considerations establish that F^M and F are continuously differentiable on $(0, \bar{x})$. □

Proof of Lemma 2. While the Lipschitz condition fails at 0, it is still possible to show that a unique solution exists by showing that there exists only one possible $\bar{x}$ where $F(\bar{x}) = 1$, and then using this as initial condition, completing the solution at zero by continuity.

By contradiction, suppose there are two distinct upper bounds $\bar{x}_1$ and $\bar{x}_2$, with $\bar{x}_2 > \bar{x}_1$. Let F_2 and F_1 be the appropriate unique solutions, respectively. Then, since both F_2 and F_1 must be strictly increasing, we have $F_1(\bar{x}_1) > F_2(\bar{x}_1)$. At any $x \leq \bar{x}_1$ for which $F_1(x) > F_2(x)$, the differential equation (20) implies $F_1'(x) < F_2'(x)$. Therefore, the difference $F_1(x) - F_2(x)$ is decreasing in x and it is strictly positive at $\bar{x}_1$, which rules out the possibility that both F_1 and F_2 satisfy the same initial condition at zero, thus establishing a contradiction.

To conclude the verification that the proposed strategies constitute an equilibrium, in addition to the argument in the main text we must check that no inactive player has a profitable deviation. Let $H^M(x) \equiv (F(x))^m$ denote the cdf of the maximum effort of the m active players on group 1; again let $G^M(x)) \equiv (G(x))^n$ denote cdf of the maximum effort of the n active players on group 2 (and let g^M be the associated density). In this situation, an inactive player on group 1 choosing effort $x \in (0, \bar{x}]$ would earn payoff

$$V_1(x) = -x + v_1 \left[H^M(x) G^M(x) + \int_x^{\infty} G^M(y)\, dH^M(y) \right],$$

[24]See Theorem 7.32 in Apostol (1974).

yielding

$$\begin{aligned}
V_1'(x) &= -1 + v_1 H^M(x) g^M(x) \\
&= -1 + v_1 (F(x))^m n (G(x))^{n-1} G'(x) \\
&= -1 + v_1 F(x) \Big[n (F(x))^{m-1} (G(x))^{n-1} G'(x) \Big] \\
&= -1 + v_1 F(x)(1/v_1) \qquad \text{(by (16))} \\
&= -(1 - F(x)) \\
&< 0 \quad \forall x \in (0, \bar{x}).
\end{aligned}$$

Thus, an inactive player on group 1 would not choose to become active. Similarly, group 2 inactive players would not become active. It follows that the F and G derived above constitute an equilibrium with m active group-1 players, n active group-2 players, and all other players inactive. □

Proof of Proposition 3. We begin by recalling the differential equation for the cdf of group 1's best-shot effort:

$$1 = v_2 \left[1 - \frac{m v_2}{n v_1} \left(1 - \Big[H(x) \Big]^{1/m} \right) \right]^{n-1} H'(x), \quad \text{with} \quad H(0) = 0. \tag{21}$$

It is simpler to work with the inverse function of the best-shot cdf, rather than H itself. To this end, we define $\eta \equiv H^{-1}$: $\eta(H(x)) = x$ for all $x \geq 0$. Then $\eta'(H(x))H'(x) \equiv 1$, so, letting $y = H(x)$, we may rewrite (21) as

$$\eta'(y) = v_2 \left[1 - \frac{m v_2}{n v_1} \left(1 - y^{1/m} \right) \right]^{n-1} \quad \text{with} \quad \eta(0) = 0. \tag{36}$$

First consider increasing v_1. In (36) the term in the square brackets is clearly increasing in v_1. Therefore, if $\tilde{v}_1 > v_1$, with corresponding inverse best-shot cdfs $\tilde{\eta}$ and η, then $\tilde{\eta}'(y) > \eta'(y)$ for all $y \in [0, 1)$. Together with $\tilde{\eta}(0) = \eta(0) = 0$, it follows that $\tilde{\eta}(y) > \eta(y)$ for all $y \in [0, 1]$. The upper end of the support of $\tilde{H}$ is higher than that for H because $\tilde{\bar{x}} = \tilde{\eta}(1) > \eta(1) = \bar{x}$. Also, suppose $x = \eta(t)$. Because $\tilde{\eta}(t) > x$ there exists $\tilde{t} < t$ such that $\tilde{\eta}(\tilde{t})) = x$. Now we see $\tilde{H}(x) = \tilde{H}(\tilde{\eta}(\tilde{t})) = \tilde{t} < t = H(\eta(t)) = H(x)$, that is, $\tilde{H}(x) < H(x)$ for any x such that $H(x) \in (0, 1)$; thus, the cdf $\tilde{H}$ first-order stochastically dominates H. Since the number of players in group 1 does not change, it must be that individual group-1 efforts are also higher in the sense of first-order stochastic dominance: $\tilde{F}(x) = (\tilde{H}(x))^{1/m} < (H(x))^{1/m} = F(x)$ for any x such that $\tilde{H}(x) \in (0, 1)$.

Next consider reducing m. We show that the term in square brackets of (21) decreases with increases in m. To see this most simply, define $\varphi(m) = m(1 - t^{1/m})$, where $t \in (0, 1)$ takes the place of $H(x)$. Then $\varphi'(m) = 1 - t^{1/m} + t^{1/m} \log\left(t^{1/m}\right) > 0$ for all $t^{1/m} \in (0, 1)$.[25] Therefore, an increase in m reduces the full

[25]Define $\psi(s) = 1 - s + s\log(s)$, for $s \in (0, 1]$. Then $\psi(1) = 0$ and $\psi'(s) = \log(s) < 0$ for any $s \in (0, 1)$. It follows that

expression in square brackets. Conversely, reducing m increases this term in square brackets, so if $\tilde{m} < m$ (and, starting with $v_1/m \geq v_2/n$) then arguing as in the previous paragraph, we conclude that $\tilde{\bar{x}} > \bar{x}$ and the cdf $\tilde{H}$ first-order stochastically dominates H. Since reducing the number of players stochastically increases efforts, it must be that the fewer active players, acting symmetrically, individually increase their efforts in the sense of first-order stochastic dominance: $\tilde{F}(x) = (\tilde{H}(x))^{1/\tilde{m}} < (\tilde{H}(x))^{1/m} < (H(x))^{1/m} = F(x)$ for any x such that $\tilde{H}(x) \in (0,1)$. □

Proof of Proposition 4. It only remains to establish the effect of increasing n. In (22), denote $\frac{mv_2}{v_1}\left(1 - z^{1/m}\right)$ by a. Notice that since by assumption $\frac{mv_2}{nv_1} \leq 1$ and $1 - z^{1/m} < 1$ for all $0 < z < 1$, it follows that $0 < a < n$. We show that the integrand of (22), now written as $A_n \equiv \left(1 - \frac{a}{n}\right)^n$, is increasing in n. Let $\tilde{A}_n$ be the extension of A_n to n continuous; then $\tilde{A}_n$ is differentiable in n and

$$\frac{\partial \tilde{A}_n}{\partial n} = \left(1 - \frac{a}{n}\right)^n \left[\log\left(1 - \frac{a}{n}\right) + \frac{a}{n-a}\right].$$

Denoting $\frac{n-a}{n}$ by t, we have

$$\frac{\partial \tilde{A}_n}{\partial n} = t^n \left[\log t - 1 + \frac{1}{t}\right].$$

Since $\log t - 1 + \frac{1}{t} > 0$ for all $0 < t < 1$, it follows that $\frac{\partial \tilde{A}_n}{\partial n} > 0$. Therefore, the integrand in (22) is increasing in n, implying that the probability that group 1 wins is increasing in n. □

Proof of Proposition 5. An active group 1 player choosing effort $\bar{x}$ earns payoff $v_1 - \bar{x}$. That same player choosing effort 0 expects his group to win with probability $\frac{m-1}{m+n-1}$, since a total of $m + n - 1$ players are choosing according to the same equilibrium distribution, and $m - 1$ of these are in group 1. Therefore,

$$v_1 - \bar{x} = \left(\frac{m-1}{m+n-1}\right) v_1,$$

which implies $\bar{x} = \left(\frac{nv_1}{m+n-1}\right) = \left(\frac{mv_2}{m+n-1}\right)$, where the second equality follows from the assumption that $mv_2 = nv_1$.

From (20) we obtain for the common equilibrium strategy F the condition that

$$\begin{aligned}\frac{1}{mv_2} &= (F(x))^{m+n-2} f(x) \\ &= \left(\frac{1}{m+n-1}\right) \frac{d}{dx} (F(x))^{m+n-1},\end{aligned}$$

from which we obtain $F(x) = \left[\left(\frac{m+n-1}{mv_2}\right) x\right]^{\frac{1}{m+n-1}}$.

$\psi(s) > 0$ for any $s \in (0,1)$. Observe that $\varphi'(m) = \psi(t^{1/m})$.

Because all players use the same strategy, each player is equally likely to be the "best shot." Therefore,

$$\Pr(\text{group 1 wins}) = \frac{m}{m+n} = \frac{\frac{v_1}{v_2} \times n}{\frac{v_1}{v_2} \times n + n} = \frac{v_1}{v_1 + v_2}.$$

□

Proof of Proposition 6. Part (i). Because $\left(\frac{v_2}{v_1}\right)\left(\frac{m}{n}\right) < 1$, (19) yields $G(0) > 0 = F(0)$, implying $G^{\max}(0) > F^{\max}(0)$. It only remains to show that $(G(x))^n > (F(x))^m$ for all $x \in (0, \bar{x})$. Using (19), this is equivalent to showing

$$\left(\left[1 - \left(\frac{v_2}{v_1}\right)\frac{m}{n}\right] + \left(\frac{v_2}{v_1}\right)\frac{m}{n}F(x)\right)^n > (F(x))^m \qquad \forall x \in (0, \bar{x}),$$

which, after taking logs, is equivalent to showing

$$\log\left(\left[1 - \left(\frac{v_2}{v_1}\right)\frac{m}{n}\right] + \left(\frac{v_2}{v_1}\right)\frac{m}{n}F(x)\right) > \frac{m}{n}\log(\,F(x)\,)\,.$$

And this inequality follows from the strict concavity of $\log(\,\cdot\,)$: for any $x \in (0, \bar{x})$,

$$\begin{aligned}
\log\left(\left[1 - \left(\frac{v_2}{v_1}\right)\frac{m}{n}\right] + \left(\frac{v_2}{v_1}\right)\frac{m}{n}F(x)\right) &> \left[1 - \left(\frac{v_2}{v_1}\right)\frac{m}{n}\right]\log(\,1\,) + \left(\frac{v_2}{v_1}\right)\frac{m}{n}\log(\,F(x)\,) \\
&= \left(\frac{v_2}{v_1}\right)\frac{m}{n}\log(\,F(x)\,) \\
&\geq \frac{m}{n}\log(\,F(x)\,)\,,
\end{aligned}$$

where the last inequality follows because $\log(\,F(x)\,) < 0$ and in part (i) we assume $v_1 \geq v_2$.

Part (ii). As in part (i), $G^{\max}(0) > F^{\max}(0)$. Next, consider x in a left neighborhood of $\bar{x}$. Letting f denote the density corresponding to F, we see, by taking limits in (17), that $f(\bar{x}) = 1/(mv_2)$. Moreover,

$$\begin{aligned}
\lim_{x \to \bar{x}} \frac{dG^{\max}(x)}{dx} &= \lim_{x \to \bar{x}} n\,(G(x))^{n-1}\,G'(x) \\
&= \lim_{x \to \bar{x}} nG(x)^{n-1}\left(\frac{v_2}{v_1}\right)\left(\frac{m}{n}\right)f(x) && \text{(by (18))} \\
&= \frac{1}{v_1} \\
&> \frac{1}{v_2} && \text{(because } v_1 < v_2\text{)} \\
&= \lim_{x \to \bar{x}} m\,(F(x))^{m-1}\,f(x) \\
&= \lim_{x \to \bar{x}} \frac{dF^{\max}(x)}{dx},
\end{aligned}$$

showing that at $\bar{x}$, $G^{\max}$ increases strictly faster than does $F^{\max}$. Therefore, for some $\varepsilon > 0$, $G(x) < F(x)$ for all $x \in (\bar{x} - \varepsilon, \bar{x})$. So near 0, $G > F$; and near $\bar{x}$, $G < F$. Thus, F and G cannot be ranked by first-order stochastic dominance. □

Construction of Example 4. **Part i):** $v_3 \in [3v/4, v]$. Suppose groups 1 and 2 have 2 active members, group 3 has 1. Then we have the following probabilities of winning when a player in the named group exerts effort x:

$$\begin{aligned}
\Pr(\text{group 1 wins} \mid x) &= F_1(x)(F_2(x))^2 F_3(x) + \int_x^{\bar{x}} (F_2(y))^2 F_3(y) f_1(y)\, dy, \\
\Pr(\text{group 2 wins} \mid x) &= (F_1(x))^2 F_2(x) F_3(x) + \int_x^{\bar{x}} (F_1(y))^2 F_3(y) f_2(y)\, dy, \text{ and} \\
\Pr(\text{group 3 wins} \mid x) &= (F_1(x))^2 (F_2(x))^2.
\end{aligned} \tag{37}$$

We look for an equilibrium that is symmetric across groups 1 and 2, with $F_1 = F_2 = F$ and associated density f. The group-3 player earns payoff $v_3 - \bar{x}$, so, for any $x \in (0, \bar{x}]$, the group-3 player's payoff is

$$v_3 - \bar{x} = V_3(x) = -x + v_3(F(x))^4,$$

implying

$$F(x) = \left(\frac{v_3 - \bar{x} + x}{v_3}\right)^{1/4} \qquad \forall x \in (0, \bar{x}]. \tag{38}$$

By right-continuity of cdfs, formula (38) also holds at $x = 0$. With symmetry and (37), we see an active group-1 player choosing effort x obtains payoff

$$V_1(x) = -x + v\left[(F(x))^3 F_3(x) + \int_x^{\bar{x}} (F(y))^2 F_3(y) f(y)\, dy\right].$$

Constancy of player 1's payoff over the randomization interval yields

$$0 = V_1'(x) = -1 + v\Big[2(F(x))^2 F_3(x) f(x) + (F(x))^3 f_3(x)\Big],$$

which can be rewritten as

$$\begin{aligned}
\frac{1}{vF(x)} &= 2F(x)F_3(x)f(x) + (F(x))^2 f_3(x) \\
&= \frac{d}{dx}\left[(F(x))^2 F_3(x)\right].
\end{aligned} \tag{39}$$

If the group-3 player's payoff is positive, then it must be that F puts an atom at 0 and F_3 does not. Therefore, upon integration, the previous differential equation yields

$$(F(x))^2 F_3(x) = \int_0^x \frac{(v_3)^{1/4}}{v}(v_3 - \bar{x} + y)^{-1/4}\, dy$$
$$= \frac{4(v_3)^{1/4}}{3v}\left[(v_3 - \bar{x} + x)^{3/4} - (v_3 - \bar{x})^{3/4}\right].$$

Substituting for $F(x)$ from (38) and solving for F_3, we obtain

$$F_3(x) = \frac{4(v_3)^{3/4}}{3v}(v_3 - \bar{x} + x)^{1/4}\left[1 - \left(\frac{v_3 - \bar{x}}{v_3 - \bar{x} + x}\right)^{3/4}\right] \quad \forall x \in [0, \bar{x}].$$

Now to find $\bar{x}$ we solve $F_3(\bar{x}) = 1$, a condition that we rewrite as

$$\frac{1}{v_3}\left(v_3 - \frac{3v}{4}\right) = \left(\frac{v_3 - \bar{x}}{v_3}\right)^{3/4}. \tag{40}$$

Equilibrium requires $\bar{x} \leq v_3$, so by (40) the above analysis will be valid if and only if $v_3 \in [3v/4, v]$. So for $v_3 \in [3v/4, v]$ we calculate

$$\bar{x} = v_3 - \frac{2^{1/3}}{8v_3^{1/3}}\left(4v_3 - 3v\right)^{4/3}.$$

So we see the foregoing is valid (*i.e.*, yields $\bar{x} \leq v_3$) if and only if $v_3 \in \left[\frac{3}{4}v, v\right]$.

Part ii): $v_3 \in [2v/3, 3v/4)$. We saw from part i that player 3's payoff is zero when $v_3 = 3v/4$. Hence, for smaller v_3 player 3's expected payoff will also be zero. Therefore, for $v_3 \in (2v/3, 3v/4)$ player 3 will choose to participate in the contest but only earns a payoff of zero. Correspondingly, equilibrium will require $\bar{x} = v_3$. Surprisingly, it turns out that in equilibrium player 3 will not randomize over the entire interval $[0, v_3]$. Rather, we look for a strategy where groups 1 and 2 randomize over $[0, v_3]$, but group 3 continuously randomizes only over $[b, v_3]$ and puts all remaining mass on 0. The value for b is not arbitrary, but will be (uniquely) determined as part of the equilibrium.

On $[b, v_3]$ the condition that group 3 player's payoff is 0 implies groups 1 and 2 use strategy

$$F(x) = \left(\frac{x}{v_3}\right)^{1/4} \quad \forall x \in [b, v_3]. \tag{41}$$

To find F_3 on $[b, v_3]$, we integrate the differential equation (39) *down* from $\bar{x} = v_3$, using the boundary condition $F(v_3) = F_3(v_3) = 1$:

$$1 - (F(x))^2 F_3(x) = \int_x^{v_3} \frac{d}{dy}\left[(F(y))^2 F_3(y)\right] dy \tag{42}$$

$$= \int_x^{v_3} \frac{1}{v}(v_3)^{1/4} y^{-1/4}\, dy$$

$$= \frac{4}{3v} v_3^{1/4}\left(v_3^{3/4} - x^{3/4}\right). \tag{43}$$

And the formula for F_3 will be derived from (42)–(43) and (41)

$$F_3(x) = \sqrt{\frac{v_3}{x}}\left(1 - \frac{4v_3}{3v} + \frac{4v_3^{1/4}}{3v} x^{3/4}\right) \quad \forall x \in [b, v_3].$$

Now consider strategies over $(0, b)$. Here group 3 places no mass on any x in this range, so $F_3(x) = F_3(b)$ (this is the size of the atom on 0). An active group-1 player choosing $x \in (0, b)$ has payoff

$$V_1(x) = -x + v\left[(F(x))^3 F_3(b) + \int_x^b (F(y))^2 F_3(b) f(y)\, dy + \int_b^{v_3} (F(y))^2 F_3(y) f(y)\, dy\right].$$

Constancy of V_1 over $(0, b)$ yields

$$0 = V_1'(x) = -1 + 2vF_3(b)(F(x))^2 f(x) = -1 + \frac{2vF_3(b)}{3}\frac{d}{dx}(F(x))^3,$$

which we integrate to find, using the boundary condition $F(0) = 0$,[26]

$$F(x) = \left(\frac{3x}{2vF_3(b)}\right)^{1/3}. \tag{44}$$

Next we use continuity of F at b:

$$\left(\frac{b}{v_3}\right)^{1/4} = F(b) = \left(\frac{3b}{2vF_3(b)}\right)^{1/3}.$$

Cubing both sides of the previous equation we find

$$F_3(b) = \frac{3}{2} \times \frac{b^{1/4}(v_3)^{3/4}}{v}. \tag{45}$$

[26]This boundary condition is necessary to guarantee the group-3 player's payoff is 0.

And continuity requires this $F_3(b)$ must agree with what we found for F_3 on $[b, v_3]$:

$$\begin{aligned}
\frac{4}{3v} v_3^{1/4}\left(v_3^{3/4} - b^{3/4}\right) &= 1 - (F(b))^2 F_3(b) && \text{(by (42) and (43))} \\
&= 1 - \left(\frac{b}{v_3}\right)^{1/2} \times \frac{3}{2}\,\frac{b^{1/4}(v_3)^{3/4}}{v} && \text{(by (44) and (45))} \\
&= 1 - \frac{3(v_3)^{1/4}}{2v} b^{3/4},
\end{aligned}$$

which yields

$$b^{3/4} = \frac{6v}{(v_3)^{1/4}}\left(1 - \frac{4v_3}{3v}\right),$$

which is strictly positive if and only if $v_3 < 3v/4$. Thus, the candidate b, given v_3, is uniquely determined as

$$b = \left[\frac{6v}{(v_3)^{1/4}}\left(1 - \frac{4v_3}{3v}\right)\right]^{4/3}. \tag{46}$$

Our candidate equilibrium is as follows:

$$F(x) = \begin{cases} \left(\dfrac{x}{v_3}\right)^{1/4} & x \in [b, v_3] \\[2ex] \dfrac{x^{1/3}}{v_3^{1/4} b^{1/12}} & x \in [0, b) \end{cases} \tag{47}$$

and

$$F_3(x) = \begin{cases} \sqrt{\dfrac{v_3}{x}}\left(1 - \dfrac{4v_3}{3v} + \dfrac{4v_3^{1/4}}{3v} x^{3/4}\right) & x \in [b, v_3] \\[2ex] \dfrac{3}{v}\left(\dfrac{v_3}{2}\right)^{2/3} (3v - 4v_3)^{1/3} & x \in [0, b), \end{cases} \tag{48}$$

where b is given by (46). The formula for b equals 0 when $v_3 = 3v/4$ (consistent with part i, where player 3 uses no atoms) and equals $2v/3$ when $v_3 = 2v/3$; in the limit as v_3 approaches $2v/3$ from above, group three effectively drops out—$F(b)$ goes to 1 as v_3 goes to $2v/3$.

It only remains to verify that player 3 does not strictly want to choose an $x \in (0, b)$. The payoff to player 3 of contributing $x \in [0, b]$ is $v_3(F(x))^4 - x$, where F is described in (47). This payoff is strictly convex on $[0, b]$, so on $[0, b]$ it is maximized at 0 or b. At $x = 0$ the payoff is zero; at $x = b$, this payoff is also zero, so, contributing $x \in [0, b]$ is not a profitable deviation. Thus, (47) and (48) constitute equilibrium strategies in part ii. □

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Table 1: The probability that group 1 wins: $v_1 = v_2$, m active group-1 members and n active group-2 members

$\downarrow m \setminus n \rightarrow$	1	2	3	4	5	6	7	8	9	10	20	50	∞
1	.5	.583333	.601852	.610156	.61488	.61793	.620062	.621638	.622848	.623809	.628037	.630504	.632121
2	-	.5	.525926	.5375	.544046	.548256	.551192	.553356	.555017	.55633	.562111	.565471	.567668
3	-	-	.5	.512165	.519126	.523624	.526768	.529089	.530872	.532286	.538503	.542123	.544492
4	-	-	-	.5	.507037	.511608	.514811	.51718	.519002	.520448	.526816	.530533	.532967
5	-	-	-	-	.5	.504586	.507807	.510192	.512029	.513486	.519918	.523677	.526141
10	-	-	-	-	-	-	-	-	-	.5	.506491	.510299	.512802
20	-	-	-	-	-	-	-	-	-	-	.5	.503816	.506327