Examples of Macroeconomic and Non-Economic Dynamic Models That are Not Self Averaging

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Abstract

This paper describes examples of non-self averaging phenomena drawn from macroeconomic and physics fields. They are models of random clusters, such as Poisson-Dirichlet models, urn models, and models of random transport through disordered media. In particular, we discuss several three-parameter extension of the two parameter Poisson-Dirichlet model. These three parameter models inherit non-self averaging asymptotic behavior of the two-parameter $PD(\alpha, \theta)$ model, with positive α. Models of random additive types and random multiplicative types are mentioned. A sufficient condition for models to be non-self averaging is also presented.

Introduction

Whether in growth or business cycle models, the fundamental reason for often complex optimization exercises is that they are expected to lead us to better understanding of dynamics of the *means* of *aggregate* variables.

The standard model thus begins with the analysis of the optimization of the representative agents, and translates the results into the analysis of the economy as a whole. Economists doing these exercises are, of course, well aware that economic agents differ, and that they are subject to idiosyncratic (or microeconomic) shocks. Their analyses are simply premised on the assumption that those microeconomic shocks and differences cancel each other out in large systems, and that the behavior of aggregate variables are represented by their means which, in turn, can be well captured by the analysis based on the representative agents.

The standard model thus explicitly or tacitly presumes the representative agent. Economic agents are homogeneous in that they face the same instantaneous probability that an "event" occurs to them. When we drop this crucial assumption, a quite different picture emerges.

It has become painfully clear that the so-called Dynamic Stochastic General Equilibrium framework is inadequate and must be substantially modified or replaced in order to deduce consequences of macroeconomic policies in economies composed of heterogeneous and random clusters of agents.

Using simple stochastic models, this paper aims to demonstrate that this tacit and yet the fundamental assumption underlying endogenous growth and real business cycle theories, namely the law of large numbers, is not generally tenable in situations with power-law tails or when groups of heterogeneous agents are involved.

We first explain the notion of "non-self-averaging", the crucial concept for our purpose. A sufficient condition for non-self averaging is later presented.

Given N data points, $X_i, i = 1, 2, \cdots {}_N$, denote the mean of these samples by

$$X_\mu = (X_1 + X_2 + \cdots + X_N)/N.$$

Then, the sample coefficient of variation of these point is

$$\hat{CV}(X) = \frac{\sqrt{\sum_{i=1}^{i=N} (X_i - X_\mu)^2}}{X_\mu}.$$

In statistical literature this ratio in the coefficient of variation is denoted by T_N, that is

$$T_N := \frac{V_N}{S_N},$$

where V_N is the sum of $(X_i - X_\mu)^2$ and $S_N = X_1 + \cdots X_N)/N$.

The sample coefficient of variation is then expressed as

$$\hat{CV}(X) = NT_N - 1.$$

In the statistical literature some analysis of the coefficients of variation are available. For example, Albrecher, and Teugels (2007) assume that $1 - F(x) \approx x^{-\rho} l(x)$, where $F(x)$ is the distribution function, $l(x)$ is slowly varying, that is, the ratio $l(\lambda x)/l(x)$ goes to 1 as x tends towards infinity, for all positive λ. Another related result is obtained by assuming that samples are drawn from normal distributions. .

These analytical results, however, require rather special assumptions such as slowly varying functions, or normal families of densities. We do not discuss these matters here, but discuss conditions under which non-self averaging behavior obtains.

A Sufficient Condition for Non-Self Averaging

The basic idea to show that a given set of random variables are non-self averaging is as follows. Let X_{mp} be the most probable value of X, and X_μ is the mean. Assume that X_{mp} ia (much) smaller than X_μ. Note that

$$E(X - X_\mu)^2 = E(X - X_{mp})^2 [1 - \frac{(X_{mp} - X_\mu)^2}{E(X - X_{mp})^2}].$$

Then, dividing both sides by X_μ^2, and noting that the ratio in the above square bracket is less than 1, we see that the left-hand side, which is the square of the coefficient of variation is positive. This means that the coefficient of variations is positive, not zero, that is the random variable X is not self -averaging. This is a sufficient condition for non-self averaging.

Note that $E(X^2) > (EX)^2$. If $E(X^2) = E(X^2)$, then X is self-averaging. See Janson (1996).

The notion of "non-self-averaging" means that a size-dependent set of random variable $X_n, n = 1, \ldots N$ of the model has the coefficient of variation which does not converge to zero as N, e.g., model size, goes to infinity.[1]

This quantity is normally expected to converge to zero as model size (e.g. the number of economic agents) goes to infinity. In this case, the model is said to be "self-averaging." Notions of averages and typical, that is, expected value and most probable value are taken to be equal in almost all macroeconomic models that are self-averaging. This presumption is important because non-self-averaging models are sample dependent, and some degree of impreciseness or dispersion remains about the time trajectories even when the number of economic agents go to infinity.

This implies that focus on the mean path behavior of macroeconomic variables does not have any scientific justification. In random products, unlike in random sums, means and most probable values are different.

In random multiplicative models expected values and most probable values diverge. We demonstrate that a common practice of taking logarithms of products to produce sums and constructing log-linear models in such situations does not work well as we will demonstrate. As pointed out by Redner

[1]This limit is called thermodynamic limit in the physics terminology.

(1990), and in Aoki (1997, App.1) this procedure of taking logarithms of products to convert products into sums and treating them as random sums involves a misapplication of the method of Laplace and produces erroneous results.

Policy Implications

The main analytical exercise by the mainstream macroeconomists such as Romer (1986), Lucas (1988), Grossman and Helpman (1991), and Aghion and Howitt (1992), is to explicitly consider the optimization by the representative agents using quadratic criterion functions in such activities as education, on-the-job training, basic scientific research, and process and product innovations. This approach is found not only in the study of economic growth, but also in research on business cycles as well.

In this paper we argue that this research program, which dominates modern macroeconomics, is misguided, because it does not recognize the distinction betweeen self-averaging and non-self averaging aspects of their optimization problems. Whether in growth or business cycle models, the fundamental motivation for often complex optimizaation excercises is that they are expected to lead us to better understanding of dynamics of the *mean* or *aggregate* variables. The standard procedure is to begin the analysis of optimization for the representative agent, and translate it into the analysis of the economy as a whole. These exercises presume that different miccroeconomic shocks and differences among agents will cancel out in the means, and the results can be well captured by the analysis on the representative agents.

We show that the phenomenon of non-self averaging has material consequences for macroeconomic policy development. Specifically, models that exhibit non-self averaging—i.e., those whose standard deviations divided by the means, do not decrease as the systems grow—are ubiquitous and macroeconomic simulaltions using them can give rise to non-informative or misleading policy results.

This phenomenon is related directly to the magnitude of economic fluctuations and is consistent with the size and scaling of fluctuations observed both recently and in the past.

By way of examples, we show how macroeconomic policy can be rendered totally ineffective solely as a result of non-self averaging. All together three types of non-self averaging models are discussed. They are two-parameter Poisson Dirichlet models; urn models; and two block of interdependent macroeconomic models. See Aoki (2008a,b,c). After a brief introduction to non-self averaging, we present analytical results on policy ineffectiveness by means of two simple introductory examples. These show the importance of coefficients of variation, and how they enter into a well-known economics problem and that if the coefficient of variation becomes large then policy becomes ineffective.

This paper then discusses relations of urn models with macroeconomic models and exhibits urn models in the literature that are non-self averaging even though this aspect is not noted in the urn literature.

In this paper we examine non-self averaging performance index, and question the role of the "mean" dynamics when the measure of approximation errors by quadratic expression does not convey useful information, either in policy multiplier context, or in conducting or designing large scale Monte Carlo studies. Models with large values of coefficients of variation have smaller policy multipliers than models with smaller coefficients of variation.

Poisson-Dirichlet model

In Feng and Hoppe (1998) and Pitman (1999) two-parameter model for stochastic clustering by agents has been invented, which is denoted by PD(α, θ). It has been interpreted that parameter α be controlling formation of new clusters, and θ growth of one of the existing clusters.

More specifically, let K_n denotes the number of clusters formed by the n entering agents who have arrived by time t. We assume that the formation of new clusters is governed by

$$\Pr(K_{n+1} = k+1 | K_1, \cdots, K_{n-1}, K_n = k) = \frac{\theta + k\alpha}{n + \theta},$$

and

$$\Pr(K_{n+1} = k | K_1, \cdots, K_{n-1}, K_n = k) = \frac{n - k\alpha}{n + \theta}.$$

It is known that this set of equations generates the Poisson-Dirichlet process. See Yamato and Sibuya (2000) for the derivation of the recursion

$$E(K_{n+1}) = \frac{\theta}{n+\theta} + (1 + \frac{\alpha}{n+\theta}) E K_n, \tag{1}$$

and from it they obtained a closed form expressions for $E(K_n)$, and its variance. Using these they have derived the asymptotic expression for K_n/n^α, and their mean and the variance. We can also calculate the expression for the coefficient of variation of K_n/n^α, although they did not focus on this important statistics. Here both parameters are assumed to be positive.

They derived that

$$E(K_n/n^\alpha) \approx \Gamma(\theta + 1)/[\alpha \Gamma(\theta + \alpha)]. \tag{2}$$

Hence $CV(K_n/n^\alpha)$ has a positive limit value as n tends toward ∞. Aoki (2008) has noted that its coefficient of variation is positive for positive values of α, and is zero when α is zero. He also devised a three-parameter version of the original two-parameter Poisson-Dirichlet model, which is described next

Three parameter extension of Poisson-Dirichlet process models

In this section we extend the two-parameter model mentioned above by introducing a third positive-valued parameter γ,with values between 0 and α, and modifying parameter θ in two ways, as shown below, and produce four ways of extension of the basic PD model.

In this section we discuss four versions of the basic three parameter models. We show that these three-parameter models are all non-self averaging. All four extensions introduced above are non-self averaging.

Version 1 Model

Consider the following set of conditional probability statements:

$$\Pr(K_{n+1} = k + 1 | K_1, \cdots, K_n = k) = \frac{\theta + k\alpha_-}{n + \theta_+},$$

$$\Pr(K_{n+1} = k | K_1, \cdots, K_n = k) = \frac{n - k\alpha_-}{n + \theta_+},$$

and

$$\Pr(K_{n+1} = k - 1 | K_1, \cdots, K_n = k) = \frac{\gamma}{n + \theta_+},$$

where $\alpha_- = \alpha - \gamma$, $\theta_+ = \theta + \gamma$, and where $0 < \gamma < \alpha$.

The newly introduced parameter γ indicates negative influences that a financial sector exerts on the new cluster formation. The third equation shows this directly by increasing the probabilty that the number of the current sector is reduced by one.

It is the main purpose of this note to show that this three parameter model is reducible two PD models.

The recursion for the expected cluster sizes is

$$EK_{n+1} = \frac{\theta_-}{n + \theta_+} + \{1 + \frac{\alpha_-}{n + \theta_+}\}E(K_n). \tag{3}$$

Now define a new random variable

$$X_n = (\theta_+/\theta_-)K_n,$$

$$E(X_1) = \theta_+/\theta_-.$$

This random variable X_n satisfies the same recursion as (1). This random variable satisfies the same recursion equation as $E(K_n)$ if α and θ are changed into θ_+ and α_-.

There are at least three other models which admit similar simple changes of variables to prove the existence of the same class of distributions as $PD(\alpha, \theta)$).

Version2

Suppose next that the conditonal probability expression is changed to have $\theta + k\alpha - \gamma$, $n - k\alpha$, and γ, all divided by $(n + \theta)$ on the right-hand side in the three conditional probability expressions.

Eq (2) is changed into

$$E(K_{n+1}) = (1 + \frac{\alpha_-}{n+\theta})EK_n + \frac{\theta}{n+\theta}.$$

Version 3

The right-hand side of the three conditional probability expressions are now $\theta + k\alpha$, $n - k\alpha - \gamma$, and γ, all divided by $(n + \theta)$.

This time the recursion for the conditional expression for K_n becomes

$$EK_{n+1} = (1 + \frac{\alpha_-}{n+\theta})EK_n + \frac{\theta_+}{n+\theta}.$$

By defining $Y_n = (\theta/\theta_+)K_n$, we obtain the same recursion as (1) except for the fact that α is replaced by α_-.

Version 4

Finally, the three conditional probabilities have $\theta + k\alpha - \gamma$, $n - k\alpha + \gamma$, and $\nu\gamma$, all divided by $\theta + n + \nu\gamma$.

In this case, let $\theta_+ = \theta + \nu\gamma$, and $\theta_- = \theta - \nu\gamma$.

Then define

$$X_n = \frac{\theta_+}{\theta_-}K_n - c,$$

where $c = \frac{\theta_+}{\theta_-}\frac{\gamma}{\alpha}$. This yields a recursion

$$EX_{n+1} = (1 + \frac{\alpha}{n+\theta_+})EX_n + \frac{\theta_+}{n+\theta_+}.$$

Then we obtain the recursion for it as the same as (2) with θ replaced with θ_+. It is straightforward to calculate the variances and means, and the coefficients of variations of these versions. The first three versions are seen to be non-self-averaging by straightforward calculations.

Since it is tedious to discuss all four versions, we discuss version 1 as an example. Its coefficient of variation is given by

$$CV_n = \sqrt{1 + \gamma/n\theta}.$$

Consequences of Non-Self Averaging Phenomenona: Some Examples

This section describes some examples of how non-self averaging disturbances reduce effects of policy multipliers. These examples hopefully go a step further in understanding my view that lack of recognizing the significance of non-self-averaging phenomena is one of the reasons why "Economists Got it So Wrong" in the latest macroeconomic crises.

Example 1: Relative Merits of Alternatives

Given a sequence of realization of a random variable X_n, $n = 1, 2, \ldots$, let $CV(X_n)$ denote their coefficients of variations. We recall that the random variable X_n is called self-averaging if the limit of $CV(X_n)$ goes to zero as n tends to infinity, and is called non-self-averaging if the limit is some positive number or tends to infinity.

If the sequence is self-averaging it is easy to see that random variables will cluster around the mean of X_n, and the expected value $E(X_n)$ may be used as representative value of the sequence. On the other hand, if the variables are not self-averaging, one can not rely on the mean to give useful information on the behavior of the random variables.

We give a simple example to illustrate this idea. Suppose that two choices are available to a set of agents. A fraction x of the agents choose alternative 1, and the rest alternative 2. Choice 1 returns $v_1(x)$ and Choice 2 returns $v_2(x)$ where x denotes the fraction of agents with Choice 1. Here we assume that the returns depend on x, that is, the fraction of agents with choice 1. Assume that policy makers can influence the agents over these choices. These two choices could be two alternate routs between two cities, or two ways of producing some goods, and so on.

For simpler explanation,suppose that the difference of the two returns

$$\delta v(x) = v_1(x) - v_2(x)$$

is a Gaussian random variable with mean $m(x)$, and variance $\phi(x)^2$.

Then, the probability that this difference is positive is given by

$$P(\delta v(x) > 0) = \frac{1}{2}[1 + \mathrm{erf}(u)],$$

where $u = m(x)/(\sqrt{(2)}\phi)$, and the error function is defined by

$$\mathrm{erf}(x) = \kappa \int e^{-u^2} du,$$

where $\kappa = 2/\sqrt{(\pi)}$, and the integral is evaluated from 0 to x.

Note that the coefficient of variation makes its appearance here. It is the ratio of the standard deviation over the mean

$$CV(x) = \frac{\phi(x)}{m(x)}.$$

We use a useful device of Ingber (1982) to approximate the error function by the hyperbolic tangent

$$\mathrm{erf}(u) \approx \tanh(\kappa u).$$

By expanding the both sides as power series in u we see that

$$\frac{1}{2}[1 + erf(u)] \approx \frac{1}{1 + \kappa u}$$

is a good approximation for small values of u, as shown by the next two expressions which compare the series expansions of the two functions

$$erf(u) = \kappa(u - \frac{u^3}{3} + \frac{u^5}{10}) + \dots,$$

and

$$tanh(u) = \kappa(u - \frac{\kappa^2}{3}u^3 + \frac{\kappa^4}{7.5}u^5) + \dots = \kappa(u - \frac{1.21}{3}u^3 + \frac{1.6}{7.5}u^5) + \cdots.$$

Using this approximation we show that the phenomenon of non-self averaging has material consequences for macroeconomic policy development in models that exhibit non-self averaging behavior are ubiquitous and macroeconomic simulations using them can give rise to uninformative or misleading policy results. See p.64 of Aoki and Yoshikawa (2007). (The third term in the error function term in the book has a typographical error. The term $u^5/5$ should be $u^5/10$.)

The phenomena of non-self averaging effects reducing effectiveness of macroeconomic policy actions are related directly to the magnitudes of economic fluctuations and are consistent with the size and scaling of fluctuations observed both recently and in the past. By means of some of these examples we show how macroeconomic policy actions can be rendered totally ineffective solely as a result of non-self averaging property of models.

Some Examples: Binary Choice Models

As an example, consider a binary choice model, McFadden (1974),or Aoki (2002, Sec. 6.3) in which each agent makes a binary choice. Here we give a simple example and leave more detailed analysis to Aoki and Yoshikawa (2007, p. 63).

Suppose that agents are faced with two choices. They have some idea of the mean return of each of the two choices and associated uncertainty by some variance expressions. Let two choices have values V_1 and V_2, but we observe it with error ϵ_i as V_i^*, $i = 1, 2$

$$V_i^* = V_i + \epsilon_i, i = 1, 2.$$

Define $\epsilon = \epsilon_1 - \epsilon_2$. Assume that ϵ is distributed as

$$Pr(\epsilon < x) = (1 + e^{-\beta x})^{-1},$$

for real number x, and where $\beta > 0$ is a parameter of this distribution.

McFadden models agents' discrete choices as the amximization of utilities U_j, $j = 1, \dots, K$, where U_j is associated with choice j and K is the total number of available choices. Let $U_j = V_j + \epsilon_j, j = 1, 2, \dots, K$. We are really interested in picking the maximum of $V's$, not of $U's$.

Policy Multiplier in Binary Choice Models

More realistically, consider a situation where a total of N agents adopt one of two production technologies, one produces $y*$ per agent, and the other y per agent per period, where $y < y^*$. The total output per period is

$$Y = ny^* + (N - n)y = N[xy^* + (1 - x)y],$$

where $x = n/N$ is the fraction of agents which chose the more efficient technology. Stochastically one of $(N - n)$ agent changes its choice at the rate

$$r = N(1 - x)\eta_1(x),$$

or one of n agents change its choice at the rate

$$l = Nx\eta_2(x),$$

where $\eta_1(x) = exp(\beta g(x))$ and $\eta_2 = 1 - \eta_1$. Here $g(x)$ is the policy multiplier. Policy maker wants to persuade agents to switch to using the high yielding technique.

In Aoki-Yoshikawa (2007, Sec. 4.2) it is shown that the parameter β is $\kappa/\phi(x), \kappa > 0$, which is a constant, and where $\phi(x)$is the standard deviation associated with this binary choice situation. Changing $g(x)$ to $g(x) + h(x)$ by policy maker haseffectively the multiplier

$$E = \delta\phi/h(\phi) = \frac{2}{1 - 2g'(\phi*)} > 0,$$

where the coefficient of variation is $\sigma(x)/g(x)$.

As the CoV gets large, the fraction x tends to 1/2, that is no policy effects in attempting to increase the output.

Many stochastic processes can be interpreted as urn models. We can easily conceive urn models for path-dependennt economic or social phenomena. See Hoppe (1984), for example.

We next discuss urn models next a ssome models exhibit non-self averaging behavior. We mention balanced triangular urn models with two colors, discussed by Puyhaubert (2003), and Flajolet et al (2006). At least some of their urn models are non-self averaging, even though their emphasis was not on this aspect.

Urn Models

Many stochastic processes can be interpreted as urn models. An important characteristic of urn models is the property of path-dependence. An obvious example would be models of contageous diseases. Balanced triangular urn mdoels with balls of two colors have been used by Puyhaubert (2003) and Flajolet et. al. (2006). Some of the urn models discussed by them are non-self averaging, even though the emphasis of their presentation was not on this aspect. Using their derivations it is a simple matter to calculate the coefficients of variation and discover that some of their models are indeed non-self averaging.

We can easily conceive urn models for path-dependent economic or social phenomena. An important characteristic of urn models is that such processes are path-dependent. An obvious example would be models of contagious diseases. See Eggenberger and Polya (1923), for an example of an urn model to describe contageous diseases. An important characteristic of urn models is that such processes are path-dependent.

Some Examples of Non-Self Averaging outside Macroeconomics

This section briefly describes models which are non-self averaging in non-economic litrature. The first is for random variables with power-law tails, and the others are comments on examples found in primarily physics literature.

Random Variables with Power-law Tails

This example was pointed out to me by Yoshi Fujiwara (his e-mail to me on Aug. 2, 2009). I thank him for his letting me have this information.

Let X has a tail distribution $P(X > x)$ which is proportional to x^{-a} for some positive a. The $m - th$ distribution of X is $N^{-1+m/a}$ for $m = 1$ and $m = 2$. Then CV^2 is proportional to $< X^2 > - < X >^2 / < X >^2 = N - 1$, hence goes to infinity as N goes to infinity, that is, in the thermodynamic limit.

Non-Self Averaging in Physics Literature

We next mention some papers in physics. In physics literature, there are several articles by Derrida (1987) on systems which exhibits non-self averaging effects. Derrida has a number of other publications related to non-self averaging. In particular, we mention statistical properties of randomly broken objects, also known as random stick braking phenomenon, and multivalley nstructures in disordered systems, Derrida and Bessis (1988), Derrida and Flyvbjerg (1987, 1989), Krapivsky, Grosse, and Ben-Nadin (2000). For example in Derrida and Flyvbjerg (1987, 1989)), statistical properties of the multivalley structure of disordered sysetm and of randomly brken objects are mentioned. Some other physics papers are Martino and Giansanti, (1998), Maslov (1993), Wiseman and Domany (1995).

Some of these papers have close resemblance with economic papers on market shares. See Sherer (1980), and Hirschman (1960) on market shares. Kawasaki and Odagaki (2003) have a model with disordered transitions which results in non-self averaging behavior.

1 Non-Self Averaging Urn Models

Many stochastic processes can be interpreted as urn models. An important characteristic of urn models is their path depedence, i.e, urn models often exhibit non-self averaging behavior. This has been noticed in models of contagious diseases modeled by urn models, Eggenberger and Polya (1932). More recently. Flajolet and Puyhaubert (2006) and Puyhaubert (2005) have examined urn models which exhibit non-self averaging behavior

A Binomial Example of Non-Self Averaging

For a binomial random variable Z with two values $z_1 > z_2 > 0$, let $Z((N)$ denote the product of N such factors, $Z(N) = z_1^n {z_2}^{N-n}$ with $n = 0, 1, \cdots N$. Then, its average value is

$$Z(N)_m = \sum C_{N,n} p^n q^{N-n} z_1^n z_2^{N-n} = (pz_1 + qz_2)^N,$$

where $q = 1 - p$, and $C_{N,n}$ is the combinatorial factor $N!/(N-n)!n!$.

Thus, the mean (average value) of the product of N binary variables is $Z(N)_m = (pz_1+qz_2)^N$. The most probable value of the product is $Z(N)_{mp} = (z_1^p z_2^q)^N$. Note that the average value of $Z(N)$ is much larger than its most probable value. Indeed this ratio

$$\frac{Z(N)_m}{Z(N_{mp}} = \frac{(pz_1 + qz_2)^N}{[(z_1)^p(z_2)^q]^N}$$

diverges exponentially in N as N tends to infinity.

A Multiplicative Growth Model: An Example

Consider a model of a sector with N firms in an economy. Firms compete with each other, and dynamics of their growth or decay are described as follows:

The total output $Y(t)$ of N firms (sectors) grows as

$$Y(t+1) = (1 + \gamma g)Y(t)$$

with probability p, or decays as

$$Y(t+1) = (1 - f)Y(t)$$

with probability $1 - p$, where p is some positive number.

The growth rate of this model of thses N firms is given by

$$r_N = \sum_{k=0}^{N} C_{N,k} p^k q^{N-k} \ln[a\frac{k}{N} + b\frac{N-k}{N}]$$

where $q = 1 - p$, and $a = 1 + \gamma g$, and $b = 1 - f$, where γ is some positive parameter to indicate interaction or externality or influence of groing furns on the rest of the firms. Note that $a > 1 > b > 0$.

The ratio k/N is the fraction of firms with positive growth, and parameter γ is to denote positive externality of the growing firms on the industry growth.

The two- and three-parameter models discussed at the beginning of this paper are basically models of random sums. The class of models of this section involves random products. It is not quite correct to dismiss random product models, saying that logarithms of random products are random sums, hence we may use log-linear approximations.

Redner (1990), among other people, showed the danger in converting random products into sums of logarithms and be content to work with ranodm sum models. As Redner shows this type of approximation often involve incorrect use of the methods of Lyapunov, see also Aoki (1996, sec. A1.) as well.

We apply the sufficient conditon derived above to the rate of growth of this multiplicative model. We calculate the mean and the most probable values of r_N and apply the sufficient condition derived above, after calculating the mean and variance. In models where means and the most probable values are not close to each other, log-linear approximations break down. By incorporating some spill-over effects among agents, we could construct random product models which exhibit "non-self averaging" behavior.

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