Revealed Preference and Stochastic Demand Correspondence: A Unified Theory

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Abstract:

We expand the theory of consumer's behavior, based on the Weak Axiom of Revealed Preference, to permit simultaneously both random choice and choice sets, which are not singletons. We provide a consistency postulate for demand behavior when such behavior is represented in terms of a stochastic demand correspondence. We show that, when the consumer spends her entire wealth, our rationality postulate is equivalent to a condition we term stochastic substitutability. This equivalence generates the following results as special cases: (i) Samuelson's Substitution Theorem, (ii) the central result in Bandyopadhyay, Dasgupta and Pattanaik (2004) and (iii) a version pertinent to deterministic demand correspondences (which independently yields Samuelson's Substitution Theorem) as alternative special cases. Relevant versions of the non-positivity of the own substitution effect and the demand theorem also follow as corollaries in each case.

Keywords: Stochastic demand correspondence, weak axiom of revealed preference, weak axiom of stochastic revealed preference, general substitution theorem.

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1. Introduction

The purpose of this paper is to extend the classical revealed preference theory of consumers' demand behavior in several directions. A central result in the classical theory is the inter-relationship between Samuelson's Weak Axiom of Revealed Preference (WARP) and the various empirically testable restrictions on the demand behavior of a competitive consumer. We extend this result by permitting simultaneously both stochastic choices and non-singleton choice sets for consumers.

In a seminal paper, Samuelson (1938) introduced his Weak Axiom of Revealed Preference (WARP) and deduced from it the conclusion that, if the consumption bundle chosen by a competitive consumer in some initial price-wealth situation costs exactly the consumer's wealth in some altered price-wealth situation, then the products of price change and quantity change for each commodity will sum to a non-negative number. This conclusion, which we shall call Samuelson's Inequality, summarizes much of the empirical content of the theory of consumers' behavior. It implies the non-negativity of the own-price substitution effect as a special case; the non-negativity of the own-price substitution effect, in its turn, yields the 'law of demand' which tells us that, other things remaining the same, a fall in the price of a normal good increases the quantity of it purchased by the consumer. Not only does WARP imply Samuelson's Inequality, but, for a consumer who always spends her entire wealth, WARP also turns out to be equivalent to Samuelson's Inequality (see Samuelson (1947) and Mas-Colell *et al*. (1995, pp. 28-32)). The equivalence of WARP and Samuelson's Inequality for a consumer who always exhausts her budget constitutes a basic result in the standard choice-based theory of consumers' behavior; for convenience, we shall term this equivalence Samuelson's Substitution Theorem.

The choice-based theory of consumer's behavior takes, as its primitive, some complete description of demand behavior for every possible price-wealth configuration. Its particular method of representation, however, imposes two major constraints at the very outset. First, faced with a given price-wealth situation, the consumer is assumed to choose a single consumption bundle. Thus, the consumer's demand behavior is constrained to representation by means of a demand *function*, rather than a demand correspondence. As a consequence of this, the analytical framework cannot, handle the choice counterpart of 'flat' indifference surfaces in the preference-based theory, even though it is not intuitively clear why, a priori, one should rule out the possibility that there may be several consumption bundles in the consumer's budget set that she considers to be equally worthy of being chosen from that budget set.[1] Second, a consumer's demand behavior is represented by means of a *unique* demand function. This assumes away the possibility of any probabilistic element in the consumer's choice. Modeling consumer's behavior by means of stochastic demand correspondences,

[1] Of course, observed market choices, the focus of Samuelson's theory, would not reveal such multiplicity of 'optimal' consumption bundles for a given budget set since, by definition, the consumer can buy only one consumption bundle belonging to her budget set. One may, however, think of the problem of a consumer's choice in terms of consumer survey experiments, where consumers are asked to choose a (possibly multi-element) subset out of some feasible set of alternatives.

instead of the traditional deterministic demand functions, would evidently allow one to accommodate these neglected aspects, thereby permitting a framework with greater flexibility and empirical scope.

The need to permit multi-element choice was noticed early. Both Arrow (1959) and Sen (1971) specified versions of WARP applicable to general choice correspondences, though they did not address the specific issue of consumers' demand behavior. Richter (1966) explicitly developed the revealed preference theory of consumers' behavior in terms of demand correspondences, rather than demand functions. Afriat (1967) and Varian (1982) developed the generalized axiom of revealed preference (GARP) to accommodate 'flat' indifference surfaces. None of these contributions however addressed the issue of expanding the central result in the traditional theory, i.e., Samuelson's Substitution Theorem, to encompass demand correspondences. Their focus instead was on identifying restrictions that would allow demand behavior to be rationalized in terms of maximization on the basis of some complete and transitive binary preference relation or even a utility function, requirements stronger than the satisfaction of WARP.[2] Thus, the primary objective of this line of enquiry was to construct a preference ordering or, possibly, a utility function from some given specification of demand behavior. It is not even clear from these contributions exactly how one should interpret either the non-positivity of the own-price substitution effect or the demand theorem, when the consumer's choice set contains multiple consumption bundles. Furthermore, this literature restricted itself to deterministic choice behavior.

Pursuing a parallel, probabilistic, line of enquiry, Bandyopadhyay, Dasgupta and Pattanaik (2004, 1999) have recently presented a rationality postulate for stochastic demand behavior, the weak axiom of stochastic revealed preference, and developed a stochastic version of Samuelson's Substitution Theorem.[3] Their result encompasses the traditional version as a special case. Their analysis, however, was carried out in terms of stochastic demand functions. Thus, while Bandyopadhyay, Dasgupta and Pattanaik (2004, 1999) departed from the classical framework by allowing the consumer to choose in a probabilistic fashion, they nevertheless constrained her to randomize only among alternative singleton sets of consumption bundles.

Integration of these different strands of analysis would thus appear to be of considerable interest. Suppose one took, as one's theoretical primitive, a representation of consumer's behavior in terms of a stochastic demand correspondence (see Section 4 below for an example of the type of observed behavior of a consumer that can be accommodated in the conceptual framework of a

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[2] Houthakker (1950) developed the strong axiom of revealed preference (SARP) precisely to bridge this gap. GARP is a generalization of SARP. Thus, much of the core empirical content of the standard theory of consumer's behavior, as summarized by Samuelson's Inequality, does not require rationalizability, just as much of the core empirical content of the traditional theory of competitive firm behavior does not entail rationalizability in terms of profit maximization (Dasgupta (2005)).

[3] Bandyopadhyay, Dasgupta and Pattanaik (2002) also apply this rationality postulate to the problem of demand aggregation, while Dasgupta and Pattanaik (2006) explore its connection with a weaker rationality postulate, viz. regularity, in a general stochastic choice context. For a recent survey, see McFadden (2005).

stochastic demand correspondence but not in the available alternative conceptual frameworks). Can one then develop a predictive theory of consumer's behavior, which expands Samuelson's WARP-based framework, to cover *simultaneously* both random choice and the choice of sets of multiple consumption bundles? Such a theory would: (i) subsume the central result of Bandyopadhyay, Dasgupta and Pattanaik (2004) as a special case, and (ii) generate a version pertinent to deterministic demand correspondences as a second special case (one which, in turn, would subsume Samuelson's Substitution Theorem as its own special case). We seek to provide such an integrated theoretical framework. We develop a rationality, or consistency, postulate for demand behavior when such behavior is represented in terms of a stochastic demand correspondence. We show that, given that the consumer spends her entire wealth with probability one, this rationality postulate turns out to be equivalent to a condition we term stochastic substitutability. This equivalence generates the following as alternative special cases: (i) Samuelson's Substitution Theorem, (ii) the central result in Bandyopadhyay, Dasgupta and Pattanaik (2004), and (iii) a version pertinent to deterministic demand correspondences, which independently yields Samuelson's Substitution Theorem. The relevant versions of the non-positivity of the own-price substitution effect and the demand theorem also fall out as corollaries in each of these special cases. Thus, our central result expands the traditional WARP-based framework to cover simultaneously both random choice of the consumer and non-singleton choice sets, thereby enhancing both its empirical scope and analytical coverage.

Section 2 presents the basic notation. Section 3 develops the idea of representing demand behavior via stochastic demand correspondences. Section 4 defines some possible properties of stochastic demand correspondences. In particular, we define our notions of a normal good and the non-positivity property of the own substitution effect in this expanded context. We also present and discuss our rationality postulate in Section 4. Our results are presented in Section 5. Section 6 concludes. Proofs are presented in the appendix.

2. Some Notation

Let $m \geq 2$ be the number of commodities, and let $M = \{1,2,...,m\}$ denote the set of commodities. $\Re_{+}, \Re_{++}$ will denote, respectively, the set of all non-negative real numbers and the set of all positive real numbers. $\Re_{+}^{m}$ is the consumption set. The elements of the consumption set will be denoted by x, x' etc. Given any consumption bundle $x \in \Re_{+}^{m}$, and given any $i \in M$, x_i will denote the amount of commodity i contained in the bundle x.

The set of all possible price vectors is $\Re_{++}^{m}$, with p, p' etc. denoting individual price vectors. For any given commodity $i \in M$, we say that two price vectors p and p' are *i-variant* iff [$p_i \neq p'_i$ and, for every commodity $j \neq i$, $p_j = p'_j$]. The set of all possible wealth levels of the consumer is $\Re_+$, with W, W' etc. denoting specific wealth levels. A *price-wealth situation* is a pair $(p, W) \in \Re^m_{++} \times \Re_+$. Thus, the set of all possible price-wealth situations is $Z \equiv \Re^m_{++} \times \Re_+$. Given a price-wealth situation (p, W), the consumer's *budget set*, $B(p, W)$, is the set $\{x \in \Re^m_+ \mid p.x \leq W\}$. For brevity, we shall typically write B, B' etc., instead of $B(p, W), B(p', W')$, etc.

Given any non-empty set T, $r(T)$ will denote the set of all possible non-empty subsets of T and $R(T)$ will denote the power set of T (thus, $R(T) \equiv r(T) \cup \{ \phi \}$, where ϕ denotes the empty set). Given two sets, T and T, $[T \setminus T']$ will denote the set of all elements of T that do not belong to T'.

3. Stochastic Demand Correspondence

The first step in our analysis is to formalize the idea of modeling a consumer's demand behavior by means of a stochastic demand correspondence, and to locate this idea in relation to other possible, more traditional, representations. Given a price-wealth situation, the flexibility we seek intuitively involves allowing the consumer: (i) to have a choice set containing multiple consumption bundles (rather than a single bundle as in the classical framework), and (ii) to do so in a (non-trivially) random fashion, so that, in contrast to the classical framework, no single choice set is assigned probability 1.

Definition 3.1.

(i) A *stochastic demand correspondence* (SDC) is a rule C which, for every $(p, W) \in Z$, specifies exactly one finitely additive probability measure Q on $(r(B), R(r(B)))$ (thus, $r(B)$ is the set of outcomes and $R(r(B))$ is the relevant algebra in $r(B)$).

(ii) A *stochastic demand function* (SDF) is a rule D which, for every $(p, W) \in Z$, specifies exactly one finitely additive probability measure q on $(B, R(B))$ (thus, B is the set of outcomes here and $R(B)$ is the relevant algebra B).

(iii) A *deterministic demand correspondence* (DDC) is a rule c which, for every $(p, W) \in Z$, specifies exactly one non-empty subset of B.

(iv) A *deterministic demand function* (DDF) is a rule d which, for every $(p, W) \in Z$, specifies exactly one element of B.

Consider any price-wealth situation (p, W). Let B denote the budget set corresponding to (p, W). Let $Q = C(p, W)$, where C is an SDC. Given the price-wealth situation (p, W), for any $A \in R(r(B))$, $Q(A)$ is the probability that the (possibly multi-element) set of chosen bundles will lie in the class *A*. $C(p,W), C(p',W')$ etc. will be denoted, respectively, by Q, Q' etc. Thus, an SDC captures the idea that, given a budget set: (i) the consumer may choose a subset with multiple elements, and (ii) she may choose among the alternative subsets available in a probabilistic fashion. An SDF, introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999), restricts the consumer to choosing a single consumption bundle, albeit allowing her to do so in a probabilistic fashion. Let $q = D(p,W)$, where *D* is an SDF. For any $A \in R(B)$, $q(A)$ is the probability that the chosen bundle will lie in the set *A*. Thus, the intuitive idea captured by an SDF is that the consumer always chooses only one consumption bundle; however, exactly which bundle is going to be chosen is determined by some probabilistic rule. A DDC, in contrast, assumes away the probabilistic element in choice, but allows the possibility that the consumer will choose multiple bundles. Thus, given a DDC, *c*, $c(p,W) \in r(B)$ is the (possibly multi-element) set of consumption bundles the consumer will choose from the budget set corresponding to (p,W). A DDF is the most restrictive, yet also the most common, form of representation of demand behavior in consumer theory. This constrains the consumer's choice from a budget set to a single consumption bundle, chosen according to some deterministic (i.e. non-probabilistic) decision rule. Given a DDF, *d*, $d(p,W) \in B$ is the bundle the consumer will choose from the budget set corresponding to the price-wealth situation (p,W).

It is evident from the preceding discussion that an SDC is the most flexible tool available for modeling a consumer's demand behavior. Intuitively, SDFs, DDCs and DDFs are all special classes of SDCs. We now proceed to provide a formal statement of this idea.

Definition 3.2.

(i) An SDC, *C*, is *singular*, iff, for all $(p,W) \in Z$, and for every $A \in R(r(B))$, $Q(A) = Q(\{A_i \in A \mid |A_i| = 1\})$.

(ii) An SDC, *C*, is *degenerate*, iff, for all $(p,W) \in Z$, there exists $A \in r(B)$ such that $Q(\{A\}) = 1$.

A singular SDC is one where the consumer's probability of choosing a set with multiple consumption bundles is zero. A degenerate SDC is one where the consumer chooses, in effect, in a deterministic fashion. A singular SDC corresponds to an SDF, a degenerate SDC corresponds to a DDC, and a singular *and* degenerate SDC corresponds to a DDF. More formally, we define the following.

Definition 3.3.

(i) An SDC, *C*, *induces* an SDF, *D*, iff, for every $(p,W)\in Z$, and for every $A\in R(B)$, $q(A)=Q(\{\{x\}\mid x\in A\})$. An SDF, *D*, *induces* an SDC, *C*, iff, for every $(p,W)\in Z$, and for every $A\in R(r(B))$, $Q(A)=q(\{x\in B\mid \{x\}\in A\})$.

(ii) An SDC, *C*, *induces* a DDC, *c*, iff, for every $(p,W)\in Z$, $c(p,W)$ is the set of consumption bundles $A\subseteq B$ such that $Q(\{A\})=1$. A DDC, *c*, *induces* an SDC, *C*, iff, for every $(p,W)\in Z$, $Q(\{c(p,W)\})=1$.

Remark 3.4. Let C be a non-singular SDC and let D be an SDF. Given that C is non-singular, we must have $Q(\{\{x\}\mid x\in B\})<1$; yet it must always be the case that $q(B)=1$. Therefore, C cannot possibly induce D. Thus, only singular SDCs can induce SDFs. Now notice that, an SDC that is induced by an SDF must satisfy $Q(\{\{x\}\mid x\in B\})=1$; thus, every SDC induced by an SDF must be singular. Notice further that every singular SDC induces some SDF, and every SDF induces some singular SDC. If a singular SDC, *C*, induces some SDF, *D*, then the singular SDC induced by *D* must be *C* itself. Analogous relations can easily be seen to hold between degenerate SDCs and DDCs, and also between DDFs and SDCs that are *both* singular and degenerate.

Remark 3.5. There is a methodological difference between the notions of DDF and SDF on the one hand and the notions of SDF and SDC on the other. The concept of choice involved in a DDF and an SDF can be interpreted in terms of the consumer's actual purchase of a consumption bundle in the market place, though, in the case of a DDF, the consumer always purchases the same consumption bundle from the same budget set, while, in the case of the SDF, the consumer's purchase from a given a budget set can be stochastic in nature. In contrast, SDFs, as well as SDCs, permit multi-element choice sets, which cannot be directly observed by watching the consumer's actual purchases in the market place: the consumer cannot simultaneously choose more than one consumption bundle in the market even when his choice set corresponding to a given budget set contains more than one consumption bundle. This implies that the axioms regarding the consumer's behavior, which are formulated in terms of possibly multi-element choice sets, are not <u>directly</u> testable when we confine ourselves to observations of the actual market behavior of the consumer. If, however, one does not confine oneself to the consumer's market choices, it may be possible to 'observe' a 'multi-element' choice set of the consumer. For example, in response to a questionnaire, the consumer can indicate that, given the price-wealth situation, (*p*, *W*), he does not mind choosing either bundle *x* or bundle *y* but he will not choose any other consumption bundle in the budget set.

Of course, a theoretical structure, constructed in terms of either a DDC or an SDC can be used to derive conclusions about the consumer's market behavior (in that case, though the observation of the consumer's market choices cannot be used to test the assumptions of the theory directly, such observations can be used to test the assumptions indirectly by directly testing the conclusions). This will require some assumption, however weak, to link the consumer's observed market behavior to his possibly multi-element choice set. As we shall argue later on, though we use the notion of an SDC as our primitive concept, our results have obvious intuitive interpretation in terms of the observed market behavior of the consumer even under the weakest conceivable assumption regarding the link between multi-element choice sets and the consumer's observed market behavior.

4. Some properties of SDCs

We now formulate some properties that an SDC may conceivably have. Our subsequent analysis in Section 5 will clarify the interconnections among these properties.

Definition 4.1. An SDC, *C*, is *tight* iff, for every $(p,W) \in Z$, $Q\left(r\left(\left\{x \in \Re_+^m \mid p.x = W\right\}\right)\right) = 1$.

Tightness is the intuitively appealing requirement that the probability of the consumer's having a choice set, such that each consumption bundle in the choice set exhausts her entire wealth, is always one. In essence, this is the counterpart of the standard global non-satiation presumption. We shall derive our substantive results in Section 5 under the assumption that the SDC satisfies this restriction.

Consider now the following scenario. Suppose, starting from some initial price-wealth situation (p,W), the price of commodity *i* falls from p_i to p_i', all other prices remaining invariant. Consider any arbitrary amount of the *i*-th commodity, α, that the consumer could possibly have bought in the initial situation. The price fall reduces the cost of every consumption bundle containing this amount of the *i*-th commodity by $(p_i - p_i')\alpha$. Suppose now that the consumer's wealth is exactly compensated for this cost reduction. Thus, suppose that the consumer's wealth is reduced by exactly $(p_i - p_i')\alpha$. Then all consumption bundles containing at least α amount of the *i*-th commodity that were initially available continue to be available in the new price-wealth situation. Furthermore, the consumer can now afford some consumption bundles containing more than α amount of the *i*-th commodity, that she initially couldn't. However, some consumption bundles containing less than α amount of the *i*-th commodity, that were initially available, now become unaffordable. How would the consumer respond to the new price-wealth situation? More specifically, what is the probability that the consumer will choose, in the new situation, a set of consumption bundles where *every* bundle contains *at least* α of the *i*-th commodity?

Suppose that this probability is not less than the initial probability of choosing a set of consumption bundles where *at least one* consumption bundle contains *at least* α of the *i*-th commodity. Suppose further that an analogous relationship holds for sets of consumption bundles containing *more than* α of the *i*-th commodity. In such a case, we shall say that the SDC satisfies *non-positivity of the own substitution effect*. Thus, in expanding this familiar notion far beyond its original classical context, we essentially: (i) impose a set-dominance criterion, and (ii) require this set-dominance criterion to be satisfied in a probabilistic fashion.

Definition 4.2. A tight SDC, *C*, satisfies *non-positivity of the own substitution effect* (NPS) iff, for every $i \in M$, every ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$, and every α, such that [p, p' are *i*-variant with $p'_i < p_i$; $\alpha \in [0, \frac{W}{p_i}]$; and $W' = W - (p_i - p'_i)\alpha$],

$$Q'(\{s' \subseteq B' \mid x'_i > \alpha \ \ for\ all\ \ x' \in s'\}) \geq Q(\{s \subseteq B \mid x_i > \alpha \ \ for\ some\ \ x \in s\}) , \tag{4.1}$$

and

$$Q'(\{s' \subseteq B' \mid x'_i \geq \alpha \ \ for\ all\ \ x' \in s'\}) \geq Q(\{s \subseteq B \mid x_i \geq \alpha \ \ for\ some\ \ x \in s\}). \tag{4.2}$$

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Our next step is to expand the familiar notion of a normal good from its classical context. As before, we shall utilize a probabilistic set-dominance criterion in order to do so. Suppose a consumer's wealth increases, while all prices remain constant. Consider any arbitrary amount of the *i*-th commodity, α. What is the probability that the consumer will choose, in the new situation, a set of consumption bundles where *every* bundle contains *at least* α of the *i*-th commodity? Suppose this probability is not less than the initial probability of choosing a set of consumption bundles where *at least one* consumption bundle contains *at least* α of the *i*-th commodity. Suppose also that an analogous restriction holds for sets of consumption bundles containing *more than* α of the *i*-th commodity. We shall then call the *i*-th commodity a normal good.

Definition 4.3. Given an SDC, *C*, a commodity, *i*, is *normal*, iff, for all $\alpha \in \Re_+$, all $p \in \Re^m_{++}$, and all $W, W' \in \Re_+$ such that $W' > W$, (4.1) and (4.2) both hold when $B = B(p,W)$ and $B' = B(p,W')$.

We shall call a good 'regular' if, intuitively, its demand does not fall with a fall in its own price, wealth and all other prices remaining invariant. As before, we shall use a probabilistic set-dominance criterion to formalize the notion of demand 'not falling'. Thus, in our subsequent analysis, the familiar demand theorem will simply constitute the claim that every normal good is also regular.

Definition 4.4. Given an SDC, *C*, a commodity, *i*, is *regular* iff, for every ordered pair $(p, p') \in \Re^m_{++}$ such that [[p, p' are *i*-variant with $p'_i < p_i$], for every $W \in \Re_+$, and for all $\alpha \in \Re_+$, (4.1) and (4.2) both hold when $B = (p, W)$ and $B' = B(p', W)$.

We now define a restriction for SDCs when more than one, possibly all, prices are allowed to change simultaneously. This condition is essentially an expansion of a condition introduced by Bandyopadhyay, Dasgupta and Pattanaik (2004) in the context of SDFs. Following their terminology, we call our condition *stochastic substitutability*. At this stage, it seems easier to motivate stochastic substitutability in terms of its instrumental value, than through any transparent intuitive interpretation. As we shall show in Section 5 below, this condition allows us to deduce a number of existing results in the theory of consumer's behavior, as special cases within a unified framework, that allows the consumer to both choose multiple consumption bundles and do so in a random fashion.

Notation 4.5. Given two price-wealth situations $(p, W), (p', W')$, let:

$$I = \{x \in \Re^m_+ \mid p.x = W \text{ and } p'.x = W'\},$$

$$G = \{x \in \Re^m_+ \mid p.x = W \text{ and } p'.x > W'\},$$

$$H = \{x \in \Re^m_+ \mid p.x = W \text{ and } p'.x < W'\},$$

$$G' = \{x \in \Re^m_+ \mid p.x > W \text{ and } p'.x = W'\},$$

$$H' = \{x \in \Re^m_+ \mid p.x < W \text{ and } p'.x = W'\},$$

$$\overline{G}' = \{s' \subseteq \{x \in \Re^m_+ \mid p'.x = W'\} \mid [s' \cap G'] \neq \phi\},$$

and

$$\overline{H} = \{s \subseteq \{x \in \Re^m_+ \mid p.x = W\} \mid [s \cap H] \neq \phi\}.$$

Definition 4.6. A tight SDC, *C*, satisfies *stochastic substitutability* (SS) iff, for every ordered pair $\langle (p, W), (p', W') \rangle \in Z \times Z$,

$$Q'(r(G')) \geq Q(\overline{H}); \tag{4.3}$$

and, for every non-empty $A \subseteq r(I)$,

$$Q'(r(G')) + Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A\}) \geq Q(\overline{H}) + Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in A\}). \tag{4.4}$$

The last major building block for our substantive analysis in Section 5 is also perhaps the central element. This is a rationality, or consistency, postulate for demand behavior. We now introduce this rationality postulate.

Definition 4.7. An SDC, C, satisfies the *weak axiom of stochastic revealed preference* (WASRP) iff, for all $(p,W),(p',W') \in Z$, and for every non-empty $A \subseteq r(B \cap B')$,

$$Q(r(B \setminus B')) + Q(\{s \subseteq B \mid (s \cap B') \in A\}) \geq Q'(\{s' \subseteq B' \mid (s' \cap B) \in A\}), \tag{4.5}$$

where $B = (p,W)$ and $B' = B(p',W')$.

WASRP imposes a consistency requirement. To see the intuitive logic underpinning this condition, consider two budget sets B, B'. Let A denote any collection of sets of consumption bundles that are available under both budget sets. Consider the collection of all subsets of B', whose overlap with B consists of some member of A. What should be the maximum probability that the consumer's chosen subset lies in this collection under B'? Every subset of B, whose overlap with B' consists of a member of the class $[r(B \cap B') \setminus A]$, continues to have that part of it available under B'. By choosing any such subset under B, the consumer, in effect, rejects all members of A. Thus, consistency appears to require that the probability mass ascribed, under B', to the collection of all subsets of B', whose overlap with B consists of some member of A, can have only two sources. These are: (i) subsets of B that have no overlap with B', and (ii) subsets of B whose overlap with B' consists of some member of A. Hence, if demand behavior is to exhibit consistency, the probability that the consumer will choose, under B', some subset whose overlap with B consists of a member of A, should not exceed the probability that the consumer's choice under B lies in one of these two classes ((i) and (ii) above). This is the nature of the restriction imposed by WASRP.

Rationality postulates analogous to our WASRP for SDCs have been developed for SDFs, DDCs and DDFs (recall Definition 3.1). We now note these specifications and clarify their connections with our rationality postulate for SDCs.

Definition 4.8.

(i) An SDF, D, satisfies the *weak axiom of stochastic revealed preference* (WASRP) iff, for all $(p,W),(p',W') \in Z$, and for every $A \subseteq (B \cap B')$,

$$q(B \setminus B') + q(A) \geq q'(A). \tag{4.6}$$

(ii) A DDC, c, satisfies the *weak axiom of revealed preference* (WARP) iff, for all $(p,W),(p',W') \in Z$, if $[c(p,W) \cap B'] \neq \phi$, then

$$c(p',W') \in [r(B' \setminus B) \cup \{s' \subseteq B' \mid (s' \cap B) = (c(p,W) \cap B')\}]. \tag{4.7}$$

(iii) A DDF, d, satisfies the *weak axiom of revealed preference* (WARP) iff, for all $(p,W),(p',W') \in Z$, if $d(p,W) \in B'$, then:

$$d(p',W') \in [(B' \setminus B) \cup \{d(p,W)\}].$$

WASRP for SDFs (Definition 4.8(i)) was introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999). This requires that the probability, under B', of the chosen consumption bundle lying in some subset of $(B \cap B')$, cannot exceed the probability, under B, of the chosen bundle either lying in that subset or being unavailable under B'. WARP for DDCs, as specified in Definition 4.8(ii), is equivalent to Richter's (1966) 'weak congruence axiom'. It is also equivalent to Sen's (1971) specification of WARP (except that Sen considers general choice problems, not the specific problem of choice by a competitive consumer). This condition requires the following. Suppose the (possibly multi-element) set of consumption bundles chosen under B has some overlap with B'. Then the set of bundles chosen under B' must either have the same overlap with B, or have no overlap with B at all. This condition is a straightforward extension of the classical WARP for DDFs. Our statement of WARP for DDFs (Definition 4.8(iii)) is equivalent to the original weak axiom of revealed preference due to Samuelson (1938). This simply requires, when the (unique) consumption bundle chosen under B is also available under B', the (unique) consumption bundle chosen under B' must either be identical to that chosen under B, or else be unavailable under B.

Remark 4.9. If a degenerate SDC, C, satisfies WASRP, then the DDC induced by C must satisfy WARP. If a singular SDC, C, satisfies WASRP, then the SDF induced by C must satisfy WASRP. If a singular and degenerate SDC, C, satisfies WASRP, then the DDF induced by C must satisfy WARP.[4]

Remark 4.10. In Definition 4.6, we introduced the concept of tightness for an SDC. Corresponding notions of tightness, intuitively implying that the consumer always exhausts her wealth in every price-wealth situation, can be easily formulated for SDFs, DDCs, and DDFs, but we do not introduce the formal definitions here.

The implications of WASRP and tightness for the SDC of a consumer are our central concern and we turn to this in the next section. Before concluding this section, however, it may be worth noting that an SDC satisfying WASRP and tightness can accommodate a wider range of market behavior than can be accommodated by any of the following: (i) a DDF satisfying WARP and tightness; (ii) a DDC satisfying WASRP and tightness; and (iii) an SDF satisfying WASRP and tightness.

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[4] The last claim is easy to check. Lemma N.1(i) and Lemma N.2(i) in the Appendix provide formal proofs of the first two claims.

Consider a world with more than two commodities. Suppose that, under the price-wealth situation $\left(p^*,W^*\right)$, the consumer's actual market purchase is observed repeatedly. She is observed to pick two bundles, x^* and y^*, with identical frequency. Similarly, under another price-wealth situation $\left(p',W'\right)$, the consumer is observed as picking two bundles, x' and y', with identical frequency. All four bundles are distinct. Furthermore, (i) $p^*x^* = p^*y^* = W^*$, (ii) $p'x' = p'y' = W'$, (iii) $p'x^* = W', p'y^* < W'$, and

(iv) $px' = W^*, py' > W^*$.

Evidently, the consumer's market purchases could not have been generated by a DDF satisfying WARP. What happens if one assumes that the consumer has an SDF satisfying WASRP? If the consumer's observed market purchases are generated by an SDF, then, given $\left(p^*,W^*\right)$, the SDF must assign probability ½ to $\{x^*\}$ and probability ½ to $\{y^*\}$. Similarly, given $\left(p',W'\right)$, the SDF must assign probability ½ to $\{x'\}$. But then the SDF violates WASRP. Hence, no SDF satisfying WASRP can generate the demand behavior specified in our example.

What if, in the spirit of Afriat (1967) and Varian (1982?????), one tries to rationalize the consumer's observed market purchases deterministically, in terms of an SDC? Intuitively, the idea then is that, given a budget set, the consumer's choice set may have several elements and that, when this happens to be the case, the consumer picks a bundle out of this multi-element set for market purchase according to some unspecified behavioral rule (essentially, some rule of thumb) since he cannot possibly purchase in the market place more than one consumption bundle belonging to the budget set. The behavioral rules for determining the actual market purchase when the choice set has multiple consumption bundles can take various forms such as: (i) "given that my choice set corresponding to $\left(p,W\right)$ is $\{x,y\}$, half of the times when the price-wealth situation is $\left(p,W\right)$, I shall purchase *x*, and the other half of the times I shall purchase *y*"; (ii) "given that my choice set corresponding to $\left(p,W\right)$ is $\{x,y\}$, 99 % of the times when the price-wealth situation is $\left(p,W\right)$, I shall purchase *x*, and the rest of the times I shall purchase *y*"; and so on (in principle, there is an infinite number of such behavioral rules). Thus, what appears to be stochastic market behavior of the consumer may really be the result of a deterministic demand correspondence together with some rules of thumb for determining the market purchases when the choice set happens to have multiple elements. The question then arises whether the observed market demand behavior described in our example above could have been generated by some tight deterministic demand correspondence satisfying WARP, together with some behavioral rules linking actual market choices to choice sets with multiple elements? If the observed market behavior in the example is generated by an SDC in this sense, then it is clear that the SDC will satisfy WARP only if the choice set defined by the SDC for (p',W') contains x^*, y^*, x' and y'. Given $p'y^* < W'$, this would mean that the SDC is not tight. Thus, no SDC satisfying WARP and tightness could have generated the observed market behavior in the example, no matter what rules the consumer might have followed to decide her market purchases when his choice set has several elements

As we have seen, the demand behavior in our example cannot be generated by either a DDF satisfying WARP and tightness or an SDF satisfying WASRP and tightness or an SDC satisfying WARP and tightness. Yet, on closer scrutiny, the demand behavior specified in our example does not appear outrageously inconsistent. Consider two demand correspondences, *a* and *b*. Under *a*, the consumer's choice set is $\{x^*, x'\}$ for each of (p^*,W^*), and (p',W'). Under *b*, the consumer's choice set is $\{y^*\}$ for (p^*,W^*) and $\{y'\}$ for (p',W'). Notice that *a* as well as *b* individually satisfies both WARP and tightness. Furthermore, the consumer's observed market purchases are compatible with randomization between *a* and *b* with equal probability if the behavioral rules that the consumer follows in the case of multi-element choice sets are such that, whenever the price-wealth situation is (p^*,W^*) and the corresponding choice set is $\{x^*, x'\}$, she always buys x^*, and, whenever the price-wealth situation is (p',W') and her corresponding choice set is $\{x^*, x'\}$, she always buys x'. Thus, in some core intuitive sense, the consumer conforms to the underlying intuition of WARP and always exhausts her wealth. Can her behavior be captured by an SDC satisfying WASRP and tightness? It can be checked that, if the consumer follows the behavioral rules mentioned above, then an SDC, which, given (p^*,W^*), assigns probability ½ to each of the choice sets, $\{x^*, x'\}$ and $\{y^*\}$, and which, given (p',W'), assigns probability ½ to each of the choice sets, $\{x^*, x'\}$ and $\{y'\}$, will satisfy WASRP and tightness and will generate the observed market purchases of the consumer.

(INDRANEEL,

AFTER THINKING FURTHER ABOUT THIS EXAMPLE, IT SEEMS TO ME THAT IT HAS SOME FEATURES, WHICH MAY CAUSE SOME DIFFICULTIES WITH THE REFEREE. FIRST, THE RULES OF THUMB ARE RATHER ARBITRARY. WHILE THINGS WORK OUT WORKS FOR THESE RULES OF THUMB, THEY WILL NOT WORK FOR OTHER MORE PLAUSIBLE RULES OF THUMB. MORE WORRYING IS SOMETHING ELSE. EVEN WITH THESE SPECIFIC BEHAVIORAL RULES, WE DO NOT REALLY KNOW WHETHER THERE EXISTS AN SDC THAT SATISFIES WASRP

AND TIGHTNESS AND GENERATES THE MARKET PURCHASES. WE HAVE NOT SPECIFIED SUCH AN SDC COMPLETELY. I AM WORRIED THAT THE REFEREE WILL PICK UP THIS GAP AND MAKE A FUSS.

5. Results

We are now ready to present our substantive results.

Proposition 5.1. (General Substitution Theorem) *A tight SDC, C, satisfies WASRP if and only if it also satisfies SS.*

Proof: See the Appendix.

The General Substitution Theorem (Proposition 5.1) is our central result. Under the assumption of tightness, it completely specifies the restrictions on demand behavior imposed by WASRP when applied to SDCs. It provides a central unifying result, in that a number of key results in the theory of consumer's behavior can be shown to stem from this result.

First notice that Proposition 5.1 yields two basic results in demand theory, non-positivity of the own substitution effect and the demand theorem, for SDCs.

Corollary 5.2. *Suppose a tight SDC, C, satisfies WASRP. Then:*

(i) *C must satisfy NPS;*

(ii) *every normal good must also be regular.*

(iii) *C is homogeneous of degree 0 in prices and wealth.*

Proof: See the Appendix.

Counterparts of these results for DDCs can also be derived from the General Substitution Theorem, as we now formally note. Since the theory of demand behavior with degenerate SDCs, i.e. DDCs, appears to have escaped attention in the literature, this case is of independent interest as well.

Corollary 5.3. *Let c be a DDC such that, for every* $(p,W)\in Z$, $c(p,W)\subseteq(\{x\in B \mid p.x=W\})$.

(i) **(Deterministic Substitution Theorem)** *c satisfies WARP iff, for every ordered pair* $\langle(p,W),(p',W')\rangle\in Z\times Z$, *such that* $p'.x\le W'$ *for some* $x\in c(p,W)$, *for all* $x'\in c(p',W')$, *and for all* $x^*\in I$,

$$[(p-p').(x^*-x')\le 0];$$ *the inequality holding strictly when either (a)* $p'.x<W'$ *for some* $x\in c(p,W)$, *or (b)* $[c(p',W')\cap I]\neq[c(p,W)\cap I]$. (5.1)

(ii) *Let c satisfy WARP. Then, for every* $i \in M$*, for every ordered pair* $\langle (p,W),(p',W') \rangle \in Z \times Z$ *such that [[* p, p' *are i-variant with* $p'_i < p_i$*], and [* $p'.x^* = W'$ *for some* $x^* \in c(p,W)$*]], and for all* $x' \in c(p',W')$,

$[x_i^* \leq x_i']$; *the inequality holding strictly when* $x_i^* < \sup\{x_i \mid x \in c(p,W)\}$.

(iii) *Let c satisfies WARP, and suppose* $i \in M$ *is normal according to the degenerate SDC induced by c. Then, for every ordered pair* $(p, p') \in \Re^m_{++}$ *such that [* p, p' *are i-variant with* $p'_i < p_i$*], for every* $W \in \Re_+$*, and for all* $x' \in c(p',W)$,

$[\sup\{x_i \mid x \in c(p,W)\} \leq x_i']$.

Proof: See the Appendix.

Notice the forms the familiar non-positivity property of the own substitution effect and the demand theorem acquire in the context of deterministic demand correspondences (Corollary 5.3(ii) and Corollary 5.3(iii)). Suppose the price of commodity i falls, all other prices remaining constant. Suppose one took any arbitrary consumption bundle that was initially chosen as the reference bundle, and reduced the consumer's wealth, so that this reference bundle cost exactly the consumer's wealth in the new situation. Then, given WARP, *no* consumption bundle chosen in the new situation can contain less of the *i*-th commodity than the amount contained in the reference bundle. If at least one bundle chosen in the initial situation contains strictly more, then *every* bundle chosen in the new situation must do so. An analogous set-dominance condition characterizes the demand theorem.

The central result of Bandyopadhyay, Dasgupta and Pattanaik (2004) also follows as a special case from Proposition 5.1, as we now specify. Recall Notation 4.5.

Corollary 5.4. (Stochastic Substitution Theorem) *Let D be an SDF such that, for every* $(p,W) \in Z$, $q(\{x \in B \mid p.x = W\}) = 1$. *D satisfies WASRP iff it also satisfies the following:*

for every ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$*, and for every* $A \subseteq I$,

$$[q'(G') + q'(A) \geq q(H) + q(A)]. \tag{5.2}$$

Proof: See the Appendix.

Bandyopadhyay, Dasgupta and Pattanaik (2004) show that the Stochastic Substitution Theorem (Corollary 5.4) generates: (i) non-positivity of the own substitution effect and the demand theorem for SDFs, (ii) equivalence of WARP with Samuelson's Inequality for DDFs (which we have termed the Samuelsonian Substitution Theorem) and (iii) non-positivity of the own substitution effect and the demand theorem for DDFs. It follows that these key results in the theory of consumer's behavior all follow as special cases of our General Substitution Theorem (Proposition 5.1). Notice that the results for DDFs, i.e. (ii) and (iii) above, can also be alternatively generated from our Corollary 5.3 as the special case where the DDC is additionally constrained to be singular. Furthermore, (i) and (iii) above, and Corollary 5.3 ((ii) and (iii)) can all be generated as special cases of Corollary 5.2.

The interconnections between these various results are summarized in Figure 1. As is evident from Figure 1, our General Substitution Theorem (Proposition 5.1) provides the core unifying result, which yields all the other results as special cases.

Insert Figure 1

Before concluding this section, we would like to comment on one aspect of our results. Our results establish properties of stochastic demand correspondences. The question arises whether, to interpret these results in terms of the observed market behavior of the consumer, we need any specific behavioral rule for linking a realized multi-element choice set and the consumption bundle that the consumer buys in the market place given that realized choice set ? The answer is that we do not need to assume any specific behavioral rule beyond the obvious rule that the purchased bundle must be a member of the realized choice set. For an example, consider our Corollary 5.2 (i), which tells us that, if a tight SDC, *C*, satisfies *WASRP*, then *C* must satisfy *NPS*. It is straightforward to interpret the property of *NPS* for the stochastic demand correspondence, *C*, in terms of observed market behavior of the consumer. Recall that *C* satisfies *NPS* if and only if, starting with any price-wealth situation (p,W), if the price of a commodity *j* falls to $p_j^{'}$ (other prices remaining the same) and, simultaneously, the consumer's wealth is reduced by $(p_j - p_j^{'})\alpha$, where α is any quantity of commodity *j* that the consumer could possibly buy in the initial situation, then the probability attached, after the fall in the price of commodity *j*, to the class of all possible choice sets, such that *every* bundle in the choice set has at least α amount of commodity *j*, is at least as great as the probability attached, in the initial situation, to the class of all choice sets, such that at least one bundle in the choice set contains at least α amount of commodity *j*. Given this definition, an obvious interpretation of *NPS* in terms of the observed market purchases is as follows: when the price of commodity *j* falls and, simultaneously, the consumer's income is reduced by $(p_j - p_j^{'})\alpha$, where α is some quantity of commodity *j* that the consumer could possibly buy in the initial situation, the frequency with which she will buy at least α amount of commodity *j* in the new price-wealth situation will be greater than or equal to the frequency with which she buys at least α amount of commodity *j* in the initial situation, and this is true no matter what behavioral rules the consumer adopts to determine which bundle she will choose to buy from any given realized multi-element choice set. Corollary 5.2 (i), therefore, has a straightforward intuitive interpretation in terms of the consumer's actually observed market behavior. Our other results have similar interpretations.

6. Conclusion

In this paper, we have expanded the standard WARP-based theory of consumer's behavior to simultaneously cover both random consumer choice and choice of multiple consumption bundles. We have offered a consistency postulate for demand behavior when such behavior is represented in terms of a stochastic demand correspondence. We have shown show that, when the consumer spends her entire wealth with probability one, our rationality postulate is equivalent to a condition we have termed stochastic substitutability. This equivalence generates: (i) the Samuelsonian Substitution Theorem, (ii) the central result in Bandyopadhyay, Dasgupta and Pattanaik (2004) and (iii) a version pertinent to deterministic demand correspondences, which independently yields the Samuelsonian Substitution Theorem, as alternative special cases. Relevant versions of the non-positivity property of the own substitution effect and the demand theorem also fall out as corollaries in every case. Thus, we have provided a core unifying result, which subsumes and expands available results. This result may perhaps be seen as providing a logical closure to the WARP-based analysis of demand behavior initiated by Samuelson (1938).

Extension of our analysis to the issue of rationalizability in terms of stochastic orderings would appear to be the natural next step. Application of our rationality postulate to the problem of aggregating demand correspondences, along the lines of Bandyopadhyay, Dasgupta and Pattanaik (2002), may constitute another useful line of investigation. We leave these issues for the future.

Appendix

Throughout the proofs, we shall use the notation presented in Notation 4.5.

Proof of Proposition 5.1.

Let the SDC, *C*, be tight.

First suppose *C* satisfies WASRP. Let $A \subseteq r(I)$. Then $[(r(H' \cup I)) \setminus A] \subseteq r(B \cap B')$; $[(r(H' \cup I)) \setminus A] \neq \phi$. Hence, by WASRP,

$$Q'(\{s' \subseteq B' \mid (s' \cap B) \in [(r(H' \cup I)) \setminus A]\})$$
$$\leq Q(\{s \subseteq B \mid (s \cap B') \in ((r(H' \cup I)) \setminus A)\}) + Q(r(B \setminus B')). \quad \text{(N1)}$$

By tightness, when $A = \phi$,

$$Q'(\{s' \subseteq B' \mid (s' \cap B) \in [(r(H' \cup I)) \setminus A]\}) = 1 - Q'(r(G')), \quad \text{(N2)}$$

$$Q(\{s \subseteq B \mid (s \cap B') \in ((r(H' \cup I)) \setminus A)\}) + Q(r(B \setminus B')) = 1 - Q(\overline{H}). \quad \text{(N3)}$$

Combining (N1)-(N3), we get (4.3).

Again, by tightness, when $A \neq \phi$,

$$Q'(\{s' \subseteq B' \mid (s' \cap B) \in [(r(H' \cup I)) \setminus A]\})$$
$$= 1 - Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A\}) - Q'(r(G')), \tag{N4}$$
$$Q(\{s \subseteq B \mid (s \cap B') \in ((r(H' \cup I)) \setminus A)\}) + Q(r(B \setminus B'))$$
$$= 1 - Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in A\}) - Q(\overline{H}). \tag{N5}$$

Combining (N1), (N4) and (N5), we get (4.4). Hence, given tightness, WASRP implies SS.

Now suppose *C* satisfies SS. Let A be any non-empty subset of $r(B \cap B')$. We define: $A_0 = A \cap r(I)$. First suppose $A_0 = r(I)$. Then, tightness and (4.3) together yield,

$$Q(r(B \setminus B')) + Q(\{s \subseteq B \mid (s \cap B') \in r(I)\}) \geq Q'(\{s' \subseteq B' \mid [s' \cap B] \neq \phi\}). \tag{N6}$$

Since

$$Q'(\{s' \subseteq B' \mid [s' \cap B] \neq \phi\}) \geq Q'(\{s' \subseteq B'\} \mid [s' \cap B] \in A),$$

and, noting that, since $r(I) \subseteq A$,

$$Q(\{s \subseteq B \mid (s \cap B') \in A\}) \geq Q(\{s \subseteq B \mid (s \cap B') \in r(I)\}),$$

it follows from (N6) that (4.5) must hold for any $A \subseteq r(B \cap B')$ such that $r(I) \subseteq A$.

Now suppose $A_0 \subset r(I)$. Then $(r(I) \setminus A_0) \neq \phi$. Hence, by SS (4.4),

$$Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in [(r(I)) \setminus A_0]\}) + Q'(r(G')) \geq$$
$$Q(\overline{H}) + Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in [r(I) \setminus A_0]\}). \tag{N7}$$

By tightness, (N7) yields:

$$Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A_0\}) + Q'(\overline{H}') \leq Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in A_0\}) + Q(r(G)),$$

where $\overline{H}' = \{s' \subseteq (G' \cup I \cup H') \mid [s' \cap H'] \neq \phi\}$. (N8)

Again, by tightness,

$$Q'(\{s' \subseteq B' \mid (s' \cap B) \in A\}) \leq Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A_0\}) + Q'(\overline{H}'). \tag{N9}$$

It follows from (N8)-(N9) that (4.5) must hold for any non-empty $A \subseteq r(B \cap B')$ such that $A_0 \subset r(I)$. Hence, SS implies WASRP, given tightness of the SDC. ◊

Proof of Corollary 5.2.

(i) Consider any ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$ such that, for some $i \in M$, [p, p' are *i*-variant with $p_i' < p_i$] and [$W' = W - (p_i - p_i')\alpha$ for some $\alpha \in [0, \frac{W}{p_i}]$]. Let the tight SDC, *C*, satisfy WASRP. Then, by Proposition 5.1, C must also satisfy SS. It is easy to check that, (i) for all $x' \in G'$, $x_i' > \alpha$, (ii) for all $x \in H$, $x_i > \alpha$, (iii) for all $x \in I$, $x_i = \alpha$, (iv) for all $x' \in H'$,

$x_i' < \alpha$, and (v) for all $x \in G$, $x_i < \alpha$. Then, noting that *C* is tight, (4.3) yields (4.1), and, putting $A = r(I)$, (4.4) yields (4.2).

(ii) Part (ii) follows from Corollary 5.2(i), Definition 4.3 and Definition 4.4.

(iii) Noting $B(p,W) = B(\lambda p, \lambda W)$ for all $(p,W) \in Z$ and all $\lambda > 0$, it can be easily checked that, if $B(p,W) \neq B(\lambda p, \lambda W)$ for some $(p,W) \in Z$ and some $\lambda > 0$, then we would have a violation of WASRP. ◊

We shall prove Corollary 5.3 via the following Lemma.

Lemma N.1.

(i) A degenerate SDC, *C*, satisfies WASRP iff the DDC induced by *C* satisfies WARP.

(ii) A tight and degenerate SDC, *C*, satisfies SS iff the DDC induced by *C* satisfies the following: for every ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$, such that $p'.x \le W'$ for some $x \in c(p,W)$, for all $x' \in c(p',W')$, and for all $x^* \in I$, (5.1) holds.

Proof of Lemma N.1. Let *C* be a degenerate SDC, and let *c* be the DDC induced by *C*. Consider any ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$. Then, for some non-empty $a \subseteq B$, $Q(\{a\}) = 1$, and for some non-empty $a' \subseteq B'$, $Q'(\{a'\}) = 1$. Furthermore, $c(p,W) = a$ and $c(p',W') = a'$.

(i) First suppose *C* satisfies WASRP, and suppose $[a \cap B'] \neq \phi$. Consider any $t \in [r(B \cap B') \setminus (a \cap B')]$. By WASRP,

$$Q(\{s \subseteq B \mid (s \cap B') = t\}) + Q(r(B \setminus B')) \ge Q'(\{s' \subseteq B' \mid (s' \cap B) = t\}).$$

Since the LHS is 0 by construction, it follows that *c* must satisfy WARP.

Now suppose *c* satisfies WARP. Notice that the degenerate SDC induced by *c* must be *C*. If $[a \cap B'] = \phi$, then evidently the requirement for WASRP is trivially satisfied. Suppose $[a \cap B'] \neq \phi$. Then, satisfaction of WARP by *c* implies: (i) for every non-empty $A \subseteq [r(B \cap B') \setminus (a \cap B')]$, the RHS of (4.5) is 0, and (ii) for $A = \{a \cap B'\}$, the LHS of (4.5) is 1. Thus, in all cases, (4.5) must hold.

(ii) Let *C* be tight. First suppose *C* satisfies SS. Suppose $p'.x \le W'$ for some $x \in c(p,W)$. By tightness of *C*, $[p.x = W]$ for every $x \in c(p,W)$. If $p'.x < W'$ for some $x \in c(p,W)$, $Q(\overline{H}) = 1$. It follows from SS (4.3) that $Q'(r(G')) = 1$. Hence, $c(p',W') \subseteq G'$. Noting Notation 4.5, the required strict inequality in (5.1) follows. Now suppose $p'.x \ge W'$ for every $x \in c(p,W)$. Since we also have $p'.x \le W'$ for some $x \in c(p,W)$, in this case $Q(\overline{H}) = 0$, whereas $(c(p,W) \cap B') = (c(p,W) \cap I) \ne \phi$. Putting $A = \{(c(p,W) \cap I)\}$, it follows from SS (4.4) that $[Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A\}) + Q'(r(G')) = 1]$. The required weak inequality in (5.1) follows. Lastly, suppose $p'.x \ge W'$ for every $x \in c(p,W)$, $p'.x \le W'$ for some $x \in c(p,W)$, and $[c(p',W') \cap I] \ne [c(p,W) \cap I]$. Putting $A = \{(c(p,W) \cap I)\}$, and noting $[Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in A\}) = 1]$ by construction, SS (4.4) then yields $Q'(r(G')) = 1$. Hence $c(p',W') \subseteq G'$. The required strict inequality in (5.1) follows.

Now suppose *c* satisfies the following: if $p'.x \le W'$ for some $x \in c(p,W)$, then, for all $x' \in c(p',W')$, and for all $x^* \in I$, (5.1) holds. Recall that the tight and degenerate SDC induced by *c* must be *C*. Notice also that, since *C* satisfies tightness, [$p.x = W$ for all $x \in c(p,W)$], and [$p'.x' = W'$ for all $x' \in c(p',W')$]. First suppose $p'.x < W'$ for some $x \in c(p,W)$. Then, by (5.1), $c(p',W') \subseteq G'$. Hence, $Q'(r(G')) = 1$. Both (4.3) and (4.4) must therefore be satisfied for all non-empty $A \subseteq r(I)$. Now suppose $p'.x \le W'$ for some $x \in c(p,W)$, and $p'.x \ge W'$ for every $x \in c(p,W)$. Then, by (5.1), $c(p',W') \subseteq [G' \cup I]$. First suppose $[c(p',W') \cap I] \ne [c(p,W) \cap I]$. Then, by (5.1), $c(p',W') \subseteq G'$. Hence, $Q'(r(G')) = 1$. Both (4.3) and (4.4) must therefore be satisfied for all non-empty $A \subseteq r(I)$. Now suppose $[c(p',W') \cap I] = [c(p,W) \cap I]$. Consider any non-empty $A \subseteq r(I)$. If $[c(p,W) \cap I] \in A$, by (5.1), $[Q'(\{s' \subseteq (G' \cup I) \mid (s' \cap I) \in A\}) + Q'(r(G')) = 1]$. If $[c(p,W) \cap I] \notin A$, $Q(\{s \subseteq (G \cup I) \mid (s \cap I) \in A\}) = 0$. Hence, in either case, (4.4) must hold. Since $Q(\overline{H}) = 0$, (4.3) must hold as well. Lastly, if $p'.x > W'$ for every $x \in c(p,W)$, then the RHS of (4.4) must be 0. Hence, (4.3)-(4.4) must trivially hold. ◊

Proof of Corollary 5.3.

(i) Suppose *c* satisfies WARP. Then, by Lemma N.1(i), the tight and degenerate SDC, *C*, induced by *c* must satisfy WASRP. By Proposition 5.1, hence, *C* must satisfy SS. Thus, by Lemma N.1(ii) the DDC induced by *C*, *c*, must satisfy (5.1) for every ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$ such that $p'.x \le W'$ for some $x \in c(p,W)$, for all $x' \in c(p',W')$, and for all $x^* \in I$. Now suppose *c* satisfies (5.1) for every ordered pair $\langle (p,W),(p',W') \rangle \in Z \times Z$ such that $p'.x \le W'$ for some $x \in c(p,W)$, for all $x' \in c(p',W')$, and for all $x^* \in I$. Then *C* must satisfy SS (Lemma N.1(ii)). Hence, by Proposition 5.1, *C* must satisfy WASRP. By Lemma N.1(i), *c* must then satisfy WARP.

Part (i) yields part (ii), while part (iii) follows from part (ii) and Definition 4.3. ◊

We shall prove Corollary 5.4 via the following Lemma.

Lemma N.2.

(i) A singular SDC, *C*, satisfies WASRP iff the SDF induced by *C* satisfies WASRP.

(ii) A tight and singular SDC, *C*, satisfies SS iff the SDF induced by *C* satisfies (5.2).

Proof of Lemma N.2. Let *C* be a singular SDC, and let *D* be the SDF induced by *C*. Then the singular SDC induced by *D* must be *C*. Consider any ordered pair $\langle (p,W),(p',W')\rangle \in Z\times Z$.

(i) First suppose *C* satisfies WASRP. For any non-empty $A\subseteq [B\cap B']$, consider $r(A)$. Since *C* is singular, and since *D* is induced by *C*, we get:

$$Q(\{s\subseteq B \mid (s\cap B')\in r(A)\})+Q(r(B\setminus B'))=q(A)+q(B\setminus B'), \quad \text{(N10)}$$

$$Q'(\{s'\subseteq B' \mid (s'\cap B)\in r(A)\})=q'(A). \quad \text{(N11)}$$

Noting that $(q'(\phi)=0)$, it follows from (N10)-(N11) that *D* must satisfy WASRP.

Now suppose *D* satisfies WASRP. Let $A\subseteq r(B\cap B')$. Let $\widetilde{A}=\{x\in (B\cap B') \mid \{x\}\in A\}$. Since *C* is singular, $Q'((\{s'\subseteq B' \mid (s'\cap B)\in A\}))=q'(\widetilde{A})$, and $[Q(\{s\subseteq B \mid (s\cap B')\in A\})+Q(r(B\setminus B'))=q(B\setminus B')+q(\widetilde{A})]$. Since *D* satisfies WASRP, noting that $\widetilde{A}\subseteq [B\cap B']$, it follows that *C* must satisfy WASRP.

(ii) Let *C* be tight. Suppose *C* satisfies SS. Consider $r(A)\subseteq r(I)$ for any non-empty $A\subseteq I$. Since *C* is singular and tight, and since *D* is the SDF induced by *C*,

$$Q'(\{s'\subseteq (G'\cup I) \mid (s'\cap I)\in r(A)\})+Q'(r(G'))=q'(G')+q'(A), \quad \text{(N12)}$$

$$Q(\overline{H})+Q(\{s\subseteq (G\cup I) \mid (s\cap I)\in r(A)\})=q(H)+q(A), \quad \text{(N13)}$$

$$Q'(r(G'))=q'(G'), \quad \text{(N14)}$$

$$Q(\overline{H})=q(H). \quad \text{(N15)}$$

Since *C* satisfies SS, it follows from (N12)-(N15) that *D* must satisfy (5.2).

Now suppose *D* satisfies (5.2). Consider any non-empty $\widetilde{A}\subseteq r(I)$. Let $a=\{x\in I \mid \{x\}\in \widetilde{A}\}$. Then, since *C* is singular,

$$Q'(r(G'))+Q'(\{s'\subseteq (G'\cup I) \mid (s'\cap I)\in \widetilde{A}\})=q'(G')+q'(a), \quad \text{(N16)}$$

$$Q(\overline{H})+Q(\{s\subseteq (G\cup I) \mid (s\cap I)\in \widetilde{A}\})=q(H)+q(a). \quad \text{(N17)}$$

Since D satisfies (5.2), it follows from (N16)-(N17) that C must satisfy (4.4). Since (5.2) implies $[q'(G') \geq q(H)]$, and C is singular, C must also satisfy (4.3). ◊

Proof of Corollary 5.4.

Suppose D satisfies: for every $(p, W) \in Z, q(\{x \in B \mid p.x = W\}) = 1$. The singular SDC, C, induced by D must then satisfy tightness. Suppose D satisfies WASRP. Then, by Lemma N.2(i), C satisfies WASRP. Hence, by Proposition 5.1, C satisfies SS. By Lemma N.2(ii), D then satisfies (5.2). Now suppose D satisfies (5.2). Then, by Lemma N.2(ii), C satisfies SS, and thus, by Proposition 5.1, WASRP. Hence, by Lemma N.2(i), D satisfies WASRP. ◊

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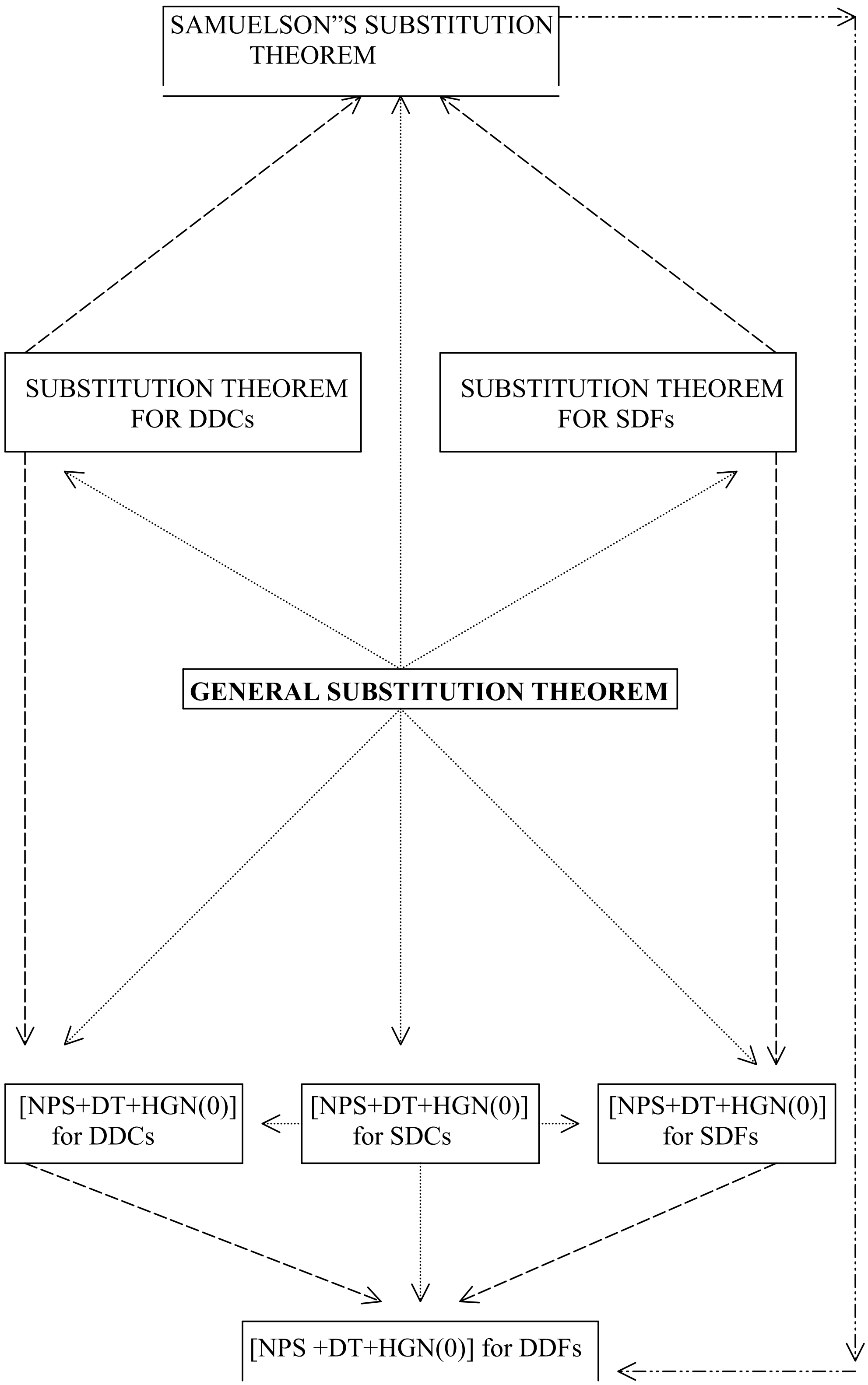
Figure 1.


SAMUELSON"S SUBSTITUTION THEOREM
SUBSTITUTION THEOREM FOR DDCs
SUBSTITUTION THEOREM FOR SDFs
GENERAL SUBSTITUTION THEOREM
[NPS+DT+HGN(0)] for DDCs
[NPS+DT+HGN(0)] for SDCs
[NPS+DT+HGN(0)] for SDFs
[NPS +DT+HGN(0)] for DDFs

(Note: DT denotes the demand theorem, and HGN(0) denotes homogeneity of degree 0 in prices and wealth.)