On a Difficulty in Welfare Economics:

Numeraire Illusion in the Marshall-Pigou-Kaldor-Hicks Methodology

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Abstract

The Kaldor-Hicks criterion (potential Pareto improvement) and the careful treatment of consumer and seller surpluses have fostered a modern revival of an older Marshall-Pigou tradition of welfare economics. That tradition was based on the parsing of a potential change into a change in the size of some "social pie" measured in money (e.g., Pigou's "national dividend") and a change in the distribution of the pie. By characterizing an increase in the size of the pie (i.e., a Kaldor-Hicks improvement) as an "increase in efficiency," this modernized Marshall-Pigou-Kaldor-Hicks (MPKH) tradition seeks to transcend the strictures of the Paretian treatment of efficiency (which would require actual compensation of the losers so that the whole change was a Pareto improvement). Economists can then with clear professional conscience make the policy recommendation for the increase in efficiency and put to one side the question of compensating the losers as a separate question of equity. However, this whole efficiency-equity analysis turns out to be vulnerable to a simple redescription of exactly the same total change using reversed numeraires. Then the "efficiency" change and the "equity" change reverse themselves so the "policy recommendation" would reverse itself as well. The flaw is the "numeraire illusion" involved in concluding that transfers in the numeraire (e.g., the compensation) do not increase value since they make no change in the size of the pie as measured by the same numeraire. Changes in a yardstick will never be revealed by *that* yardstick—but are revealed by switching to a different yardstick (or numeraire). This result undercuts the major applications of the MPKH methodology in the standard Chicago school ("social wealth" maximization) of law-and-economics, cost-benefit analysis, policy analysis, and related parts of applied welfare economics.

Key words: Kaldor-Hicks criterion, wealth maximization, efficiency, equity, and cost-benefit analysis.
JEL: D6 (welfare economics), K (law and economics).

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Introduction: Pareto versus Marshall-Pigou-Kaldor-Hicks

The Paretian revolution in normative economics established the possibility of defining efficiency (i.e., Pareto optimality wherein no one can be made better off without making someone else worse off) without using interpersonal comparisons of utility or preferences. This treatment of efficiency is often seen as a necessary condition for a maximum of "social welfare." But the notion of Pareto efficiency can also be seen as part of a rights-based approach to normative economics [e.g., Ellerman 1992, 2004] that takes seriously the differences between persons and that accordingly eschews any given[1] social scalar ("social welfare") that morally ought to be maximized. The Paretian conditions are then the necessary conditions for the vector maximization of individual welfares.

The older Marshall-Pigou tradition in the economics of welfare—initially challenged by the Paretian revolution—was based on a fundamental distinction between the size versus distribution of the "social pie" (e.g., Pigou's "production" versus "distribution" of the "national dividend" [1960]). This social pie was *not* to be identified with welfare [e.g., Pigou's "economic welfare"] since the quantity of overall social welfare could be affected by both the size *and* distribution of the pie [e.g., Pigou's "national dividend"]. The "pie" that economists would be "professionally" concerned with maximizing is an intermediate aggregate expressed in the measuring rod of money and variously known as the social or national dividend (or product), net social benefits (e.g., in cost-benefit analysis), or social wealth (e.g., in the law-and-economics literature) among other labels.[2] An increase in "efficiency" was identified with an increase in the size of that pie whereas changes in the slices of the pie was a question of "equity" outside of the scientific bailiwick of economics.

However, instead of taking the definition of efficiency in terms of vector maximization as an opportunity to explore non-welfarist approaches to normative economics (e.g., rights-based theory), the economics profession has largely bridled at the austerity and "impracticality" of the Paretian definition.[3] The rehabilitation of the Marshall-Pigou approach was lead by the introduction of the Kaldor-Hicks criterion[4] for a *potential* Pareto improvement (the winners in a proposed change could compensate the losers but don't necessarily do so) and by the modern treatment[5] of consumers' surplus. Kaldor was quite explicit about laying the groundwork to justify the older Marshall-Pigou way of thinking.

[1] Persons can always come together, negotiate, and agree on certain common goals measured by a scalar quantity (e.g., value-added in an economic enterprise) but those constructed goals are not ethically "given" independent of the common agreement.

[2] Mishan 1982 simply calls it the social "V" (as in Value) which sums the individual "vs." Some authors might muddle the social pie with social welfare. Some write loosely and interchangeably about the two notions and thus the efficiency part is pictured as an increase in welfare and the equity part as a redistribution of a given total welfare. Then they refer to the possibility of different people having different welfare weights for their share of the social pie—which requires that welfare be distinct from the pie itself.

[3] The principal exception is the catallactist school of Austrians and the constitutional economics of the Virginia neo-Austrians led by James Buchanan [e.g., 1999]. See Hicks 1975 for an interesting juxtaposition of the catallactics (exchange) approach in its Lausanne and Austrian versions with the "production and distribution of the national product" approach of the Marshall-Pigou tradition.

[4] See Kaldor 1939 and Hicks 1939 for the original articles and Mishan 1964 for a survey of the Kaldor-Hicks and later innovations.

[5] The notion of consumers' surplus [Marshall 1961] and the related notion of sellers' surplus are important tools in the Marshall-Pigou tradition so Hicks' 1941 rehabilitation of consumer's surplus using utility-compensated demand

> This argument lends justification to the procedure, adopted by Professor Pigou in *The Economics of Welfare*, of dividing "welfare economics" into two parts: the first relating to production, and the second to distribution. [Kaldor 1939, 551]

The satisfaction of the Kaldor-Hicks criterion was to be interpreted as an increase in that intermediate aggregate pie of national income or social wealth which was then taken as the less strict notion of an "efficient" change. Thus the notion of an "efficient" change was broadened from just a Pareto improvement (some people becoming better off and no one worse off, i.e., an improvement in the vector ordering of individual welfares) to a Kaldor-Hicks improvement (a potential Pareto improvement).

The Marshall-Pigou tradition was thus modernized by Kaldor and Hicks, and the seemingly austere Paretian notion of efficiency was relaxed to the "Kaldor-Hicks (wealth maximization…) concept of efficiency" [Posner 2000, 1153].[6] Today any change that increases the "social wealth" (e.g., according to the Kaldor-Hicks criterion) is routinely interpreted as an "increase in efficiency" particularly in the law-and-economics literature, cost-benefit analysis, policy analysis, and other parts of applied welfare economics [e.g., Just et al. 1982]. In general, the closer the economics is to being applied, the more it reproduces the Marshall-Pigou approach with the modern Kaldor-Hicks glosses.

The Failure of Numeraire Invariance

Redescriptions of Transactions with Reversed Numeraires

The MPKH methodology is vulnerable to a surprisingly simple countermove, namely a redescription of the same situation with a reversed numeraire. Any market or even potential exchange of a "this for that" (*quid pro quo*) has an inverted or inverse description as a "that for this" (*quo pro quid*). It is exactly the same situation, only described from an inverted perspective. Any analysis that applied economics might use to recommend social changes with winners and losers should at least be invariant to something as trivial as a redescription of the same changes with a different numeraire.[7]

Our main result is that the fundamental Marshall-Pigou-Kaldor-Hicks methodology described above is not invariant under numeraire inversion. The part of the total change first described as the "increase in size of pie" (the "project") becomes the "mere redistribution of the social pie" in the inverted description. The other part of the change (i.e., the "compensation") previously

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curves and Willig's 1976 justification of using Marshallian (uncompensated) demand curves as an approximation were important in reviving the thought patterns of that tradition.

[6] In a "lawyer's list" of reasons to use the KH criterion, some authors assert that it will lead in the long run to a Pareto improvement or that there has already been a unanimous constitutional level agreement to use the KH criterion in public policy decisions. It is hard to take these empirical assertions seriously when the authors offer no evidence. Moreover, such a "defense" of the KH criterion is intellectually incoherent since it attempts to reduce the KH condition to the Pareto or unanimity condition as if to admit that it was not an alternative after all. This paper assumes that the KH condition is taken as a genuine alternative to the Pareto-unanimity condition—as the foundation for economists to support some changes that will benefit some and hurt others on "efficiency" grounds.

[7] While money is usually taken as the numeraire, for our purposes the numeraire is only the commodity used as the unit of account in which benefits and costs are stated. The results do not depend on the numeraire having any of the other usual characteristics of money such as a store of value or a medium of exchange.

characterized as the "mere redistribution of the social pie" becomes the part that "increases the size of the social pie." Reversing numeraires reverses "project" and "compensation," i.e., reverses the "efficiency" and "equity" parts of the total change—which would reverse the so-called "efficiency recommendation". The numeraire invariant result is that both parts of the change will be a vector improvement (i.e., an actual Pareto improvement). The MPKH attempt to use "potential Pareto improvement" to go beyond the Pareto criterion thus fails.

Numeraire Illusion

The key step in going from Paretian reasoning to the MPKH reasoning was that one could have an "efficient" change by dropping out (i.e., leaving as only "potential") the changes involving only the numeraire since that "compensation" did not change the size of the pie as measured in that same numeraire. But we show that this is only what might be called *numeraire illusion*; changes in the size of a yardstick cannot be revealed by using that same yardstick. That is why the MPKH reasoning is so vulnerable to a simple redescription of the same total changes using a different numeraire as the yardstick.

Thus we conclude that the MPKH reasoning fails due to a basic interpretive error; an attribute of a specific measurement system (using metric of "money") is misinterpreted as if it were an attribute of the underlying situation being measured (numeraire illusion)—and the mistake is exposed by changing the measurement system (numeraire reversal).

Perhaps before going further, it would be useful to explicitly differentiate this critique of the MPKH methodology from the previous more fine-grained criticisms. For instance, Tibor Scitovsky [1941] pointed out certain problems in the Kaldor-Hicks criterion (i.e., the project and compensation might have such strong income effects that the KH criterion then recommended a return to the original state) and suggested a solution. As analyzed by Boadway [1974] and Blackorby and Donaldson [1990], certain other problems might arise in a general equilibrium setting to connect Hicks' compensating variation treatment of consumers' surplus (aggregate willingness-to-pay) and the Kaldor-Hicks criterion. These criticisms show that in certain theoretical cases, income and general equilibrium effects can lead to anomalies that complicate the use of the KH criterion. But those theoretical criticisms have not much slowed the application of the KH criterion in cost-benefit analysis, the "wealth maximization" school of law and economics, and other parts of applied welfare economics. In any case, those criticisms are unrelated to the numeraire illusion critique developed here which applies to all uses of the MPKH methodology, not just to anomalous special cases.

A Simple Generic Example

The MPKH methodology is the basis for the maximization of "net social benefit" in cost-benefit analysis as well as for the "social wealth" maximization at the foundation of the orthodox economic approach to law (the Chicago School of law and economics). In these contexts, it is not easy (but not impossible) to envisage a numeraire reversal so the failure of numeraire invariance in hidden from normal view. But we are looking at the underlying economic reasoning of the MPKH methodology and it can be applied to situations where numeraire inversions are trivial. Indeed such examples are in the law and economics textbooks themselves.

Consider the following simple but generic example from David Friedman's book: *Law's Order: What Economics has to do with Law and why it matters* [2000]. Mary has an apple which she values at fifty cents while John values an apple at one dollar. There might be a voluntary exchange where Mary sold the apple to John for seventy-five cents. There are two changes in that Pareto improvement: the transfer of the apple from Mary to John and the transfer of seventy-five cents from John to Mary. Let us apply social wealth maximization reasoning to the transfer of the apple using money as the numeraire. Since the apple was worth fifty cents to Mary and a dollar to John, social wealth would be increased by fifty cents by the apple transfer from Mary to John. That is an increase in efficiency. The other change, the transfer of seventy-five cents from John to Mary, is a question of distribution or equity. Social wealth (measured in dollars and cents) would be unchanged by the mere transfer of seventy-five cents from one person to another.

> It would still be an improvement, and by the same amount, if John stole the apple—price zero—or it Mary lost it and John found it. Mary is fifty cents worse off, John is a dollar better off, net gain fifty cents. All of these represent the same efficient allocation of the apple: to John, who values it more than Mary. They differ in the associated distribution of income: how much money John and Mary each end up with.
>
> Since we are measuring value in dollars it is easy to confuse "gaining value" with "getting money." But consider our example. The total amount of money never changes; we are simply shifting it from one person to another. The total quantity of goods never changes either, since we are cutting off our analysis after John gets the apple but before he eats it. Yet total value increases by fifty cents. It increases because the same apple is worth more to John than to Mary. Shifting money around does not change total value. One dollar is worth the same number of dollars to everyone: one. [Friedman 2000, 20]

Now describe exactly the same situation but from an inverted perspective with a numeraire reversal from dollars to apples. Mary was at a point where her marginal rate of substitution of dollars per apple was one-half so her marginal rate of substitution of apples per dollar would be the reciprocal, namely two apples per dollar. John's marginal rate of substitution of dollars per apple was one so its reciprocal is also one. Now apply the reasoning of social wealth maximization (measured in apples) to the proposed change of transferring seventy-five cents from John to Mary. The seventy-five cents is only worth three-fourths of an apple to John while the seventy-five cents is worth three-halves apples (2 x .75 = 1.5) to Mary. Hence the social pie (which is now an apple pie) is increased by three-fourths apples by the transfer of seventy-five cents from John to Mary. Hence *that* transfer is the efficient change (the increase in social wealth). Whether or not an apple is actually transferred from Mary to John is now a question of equity or distribution which leaves the social (apple) pie unchanged. Paraphrasing Friedman's classic statement of numeraire illusion; one apple is worth the same number of apples to everyone: one.

There has been no change in Mary's or John's preferences; exactly the same underlying situation is described first using dollars as numeraire and then using apples as numeraire. Yet the results of the social wealth maximization reasoning (and the underlying MPKH logic) changed completely between the two descriptions. The "efficiency" part and the "equity" part of the total change reversed themselves under the descriptive inversion as indicated in the following table.

Table 1. Reversal of Efficiency and Equity Parts under Numeraire Reversal

	Normal Description (money = numeraire)	**Inverse Description (apples = numeraire)**
Increase in size of "social pie"	Transfer of apple from Mary to John	Transfer of seventy-five cents from John to Mary
Redistribution of "social pie"	Transfer of seventy-five cents from John to Mary	Transfer of apple from Mary to John

The argument that a "dollar is worth the same number of dollars to everyone: one"[8] pin-points the problem that we have called "numeraire illusion." Transfers in whatever is taken as the numeraire will always seem to not change the size of the pie as measured in the same numeraire. Changes in a yardstick will never be revealed by *that* yardstick; one needs to use a different yardstick. And hence the conclusion about the "merely redistributive" part of the transfers will be vulnerable to a change in numeraire. Any transfers in the new numeraire will then appear to be merely redistributive. By reversing the numeraires, we are therefore able to reverse the efficiency-equity parsing itself.

In general, transfers of commodities between parties will benefit some and hurt others, and transfers of money (or any numeraire commodities) between parties *will do the same*. There are no objective (e.g., numeraire invariant) grounds for taking the non-numeraire transfers as a "change in social wealth" while the numeraire transfers are a "mere redistribution of existing social wealth"—although it always appears that way since "social wealth" is measured in terms of the numeraire.

The belief that there is some objective increase in "total value" in the "efficiency" part of a change satisfying the KH criterion is now quite widespread in the conventional "economic way of thinking." Many applied economists using cost-benefit analysis assume that criticism can only come from those who would emphasize different distributional weights for dollars going to the rich or to the poor, such as Boadway and Bruce [1984] and Blackorby and Donaldson [1990], in contrast to Harbinger's [1971] hard-nosed approach of "a dollar's a dollar for all that"—all *as if* the efficiency-equity parsing was numeraire invariant. But our criticism is quite different; no notion of social welfare is used. But given some notion of "social welfare," the debate over distributional weights at least makes sense. The value of a dollar to a Mary or a John is being measured by a *different* yardstick, namely social welfare. But numeraire illusion—"One dollar is

[8] This argument has the same information content (namely, zero) as the "argument" that inflation is impossible since "at any point in time, whatever commodity bundle can be purchased with a dollar will always have the same dollar value: one." It is true but it's a tautology that tells one nothing about inflation. To measure inflation, one needs a different yardstick such as a fixed standard market basket of commodities. The idea of a "different yardstick" is present in the idea of a *relative* price, the price of x in terms of (a different) y. The only price that has no information content is the "self-price" of the numeraire, one. In a similar context, Samuelson remarks that "the money-metric marginal utility of *income* is constant at unity. For how could it be otherwise? If you are measuring utility by money, it must remain constant with respect to money: a yardstick cannot change in terms of itself." [1979, 12]

worth the same number of dollars to everyone: one"—is only the tautologous assertion that a dollar has the same value to a Mary or a John *in terms of dollars* (rather than in terms of welfare). Friedman's assertion that the apple transfer increased "value" while the money transfer didn't was not a statement about distributional weights in some social welfare function; it was pure and simple numeraire illusion.

In "wealth maximization" law and economics and in cost-benefit analysis, the MPKH methodology is the same. In the law and economics literature, the apple-transfer would be some proposed change in the law. In cost-benefit analysis, the apple-transfer would be some complex project under consideration. Since it would be hard to think of a proposed legal change or a proposed project as the numeraire to redescribe the situation, the flaw in the MPKH reasoning is more hidden from view.[9] But if the underlying MPKH reasoning is blatantly flawed in a simple apples-and-dollars example, how could it suddenly become valid when the apple transfer is replaced by a legal change or by a complex project? The "ostrich defense"—not looking at cases where the numeraire can be easily reversed—will not do.[10]

The Basic Argument in Schematic Form

The controversies about measuring "consumers' surplus" or "aggregate willingness to pay" by integrating under Marshallian demand curves or Hicksian compensated demand curves are not germane to our point. Hence we will avoid those controversies by making the simple and basic point using differential changes (i.e., "small changes at the margin") around a point *independently* of any integration over a path of finite changes.

There are three commodities X, Y, and Z only two of which, X and Y, are involved in the transfers. There are two "persons" or, more abstractly, two value systems as might be given in the vector maximization of two (differentiable) functions (e.g., two utility functions):

$$[f(X_1,Y_1,Z_1),\ g(X_2,Y_2,Z_2)].$$

We consider differential changes about some given endowment point. In the value system determined by each function, the relative prices are determined by the marginal rates of substitution such as:

$$P_{1x} = MRS^1_{zx} = \frac{f_x}{f_z}$$

and similarly for the other commodities and the second value system determined by g. Let P_{1x} and P_{2x} be those prices of X in terms of Z that are "subjective" or internal to the two systems where we assume that $P_{1x} > P_{2x}$. Let P_x be an intermediate "public" price of X in terms of Z. Similarly let P_{1y} and P_{2y} be the prices of Y in terms of Z in the two systems such that $P_{1y} < P_{2y}$ and let P_y be an intermediate "public" price of Y in terms of Z. The prices of X in terms of Y can be obtained from the Z prices:

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[9] Numeraire inversion "therapy" can still be applied with complex projects. As is shown in the appendix, the key is to use a scalar scale factor for the project as the numeraire in the inverted description.

[10] Alternatively, one cannot defend one's assertion that "all cows are gray" by only looking at cows at night.

$$P_1 = P_{1x}/P_{1y} > P = P_x/P_y > P_2 = P_{2x}/P_{2y}.$$

The prices of Y in terms of X would be obtained by inverting the prices of X in terms of Y.[11]

Thus we have three arrays of prices corresponding to the three different numeraires, X, Y, and Z. In the first description of transfers dX and dY between systems or "persons" 1 and 2, Y is the numeraire. Then we redescribe the same transfers with X as the numeraire (numeraire inversion). Both these descriptions will involve numeraire illusion since the numeraire is one of the commodities involved in the transfers being evaluated. The MPKH reasoning will apply to each of these cases but give opposite results. Then we evaluate the dX and dY transfers using a non-involved commodity Z as the numeraire. Then no numeraire illusion arises and the MPKH reasoning does not apply.

We start with the description using Y as the numeraire.

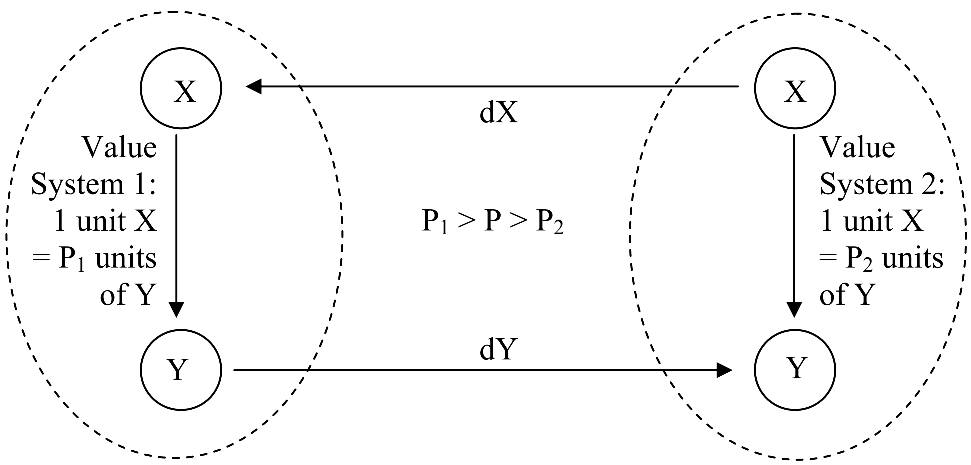

Figure 1. Two relative values for X in terms of Y

If dX is transferred from where it has a lower value in system 2 to where it has a higher value in system 1, then the Y-cost of taking dX out of 2 is P_2dX while the gain from adding dX to 1 is P_1dX. Thus the increase in Y pie from the dX transfer is:

$$\Delta Y = (P_1 - P_2)dX > 0.$$

Now suppose that $dY = PdX$ units of Y are transferred from 1 to 2. The cost to 1 is $dY = PdX$ units of Y and the gain to 2 is dY units of Y so the transfer in dY units of Y (or any other units of Y) yields no change in the size of the Y pie.[12] But there is a change in the distribution of the pie. The net change for 1 is: $(P_1 - P)dX > 0$, and the net change for 2 is: $(P - P_2)dX > 0$ so both 1 and 2 are better off and the two positive slices add up to the Y pie:

[11] Since the first value system is determined by the ratios of the "absolute" prices f_x, f_y, and f_z for the commodities and similarly for the second system, each value system is internally consistent in the sense of not allowing profitable arbitrage by trade within the system at those prices [see Ellerman 1984 for the notion of arbitrage-free relative prices]. Value-increasing trade can only occur between the systems or persons.

[12] Metaphorically, no matter how much a yardstick expands or contracts, when used to measure itself it will always record "no change."

$$(P_1 - P)dX + (P - P_2)dX = (P_1 - P_2)dX = \Delta Y.$$

The first term, $(P_1 - P)dX$, is the marginal version of X-consumer's surplus while the second term, $(P - P_2)dX$, is the marginal version of X-supplier's surplus.

So far, this is just mathematics. Then the MPKH reasoning is mislead by the numeraire illusion involved in measuring the effect of the dY transfer in terms of Y to conclude that the dY transfer added no value or wealth; it was only a redistribution. The value increase was all in the dX transfer so it can be recommended on grounds of "efficiency" while the dY transfer can be treated separately as a question of "equity."

But this asymmetric treatment of the dX and dY transfers is only a consequence of the asymmetric choice of the numeraire to evaluate the transfers. Reverse the choice of numeraires and the conclusions will be reversed. Taking X as the numeraire, $P_1' = 1/P_1$ is the price of a unit of Y in units of X in system 1 while $P_2' = 1/P_2$ is the price of a unit of Y in terms of X in 2.

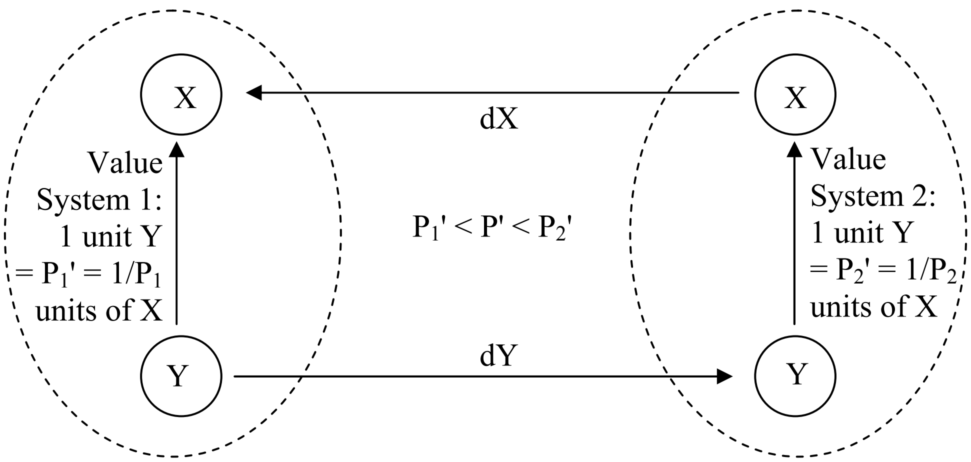

Figure 2. Two relative values for Y in terms of X

We now evaluate the results the same dY transfer from 1 to 2. The loss to 1 is $P_1'dY$ while the gain to 2 is $P_2'dY$ so noting that $P_2' > P_1'$, we have the increase in the X pie from the dY transfer as:

$$\Delta X = (P_2' - P_1')dY > 0.$$

Now taking $P' = 1/P$, suppose that $P'dY = P'PdX = dX$ units of X are transferred from 2 to 1. The cost to 2 is dX units of X and the gain to 1 is dX units of X so the transfer in dX units of X (or any other units of X) yields no change in the size of the X pie (i.e., the self-measuring yardstick records no change). But there is a change in the distribution of the pie. The net change for 2 and 1 is respectively:

$$(P_2' - P')dY > 0 \text{ and } (P' - P_1')dY > 0,$$

so both 1 and 2 are better off and the two positive slices sum to the X pie:

$$(P_2' - P')dY + (P' - P_1')dY = (P_2' - P_1')dY = \Delta X.$$

The term, $(P_2' - P')dY$, is the marginal Y-consumer's surplus while the second term, $(P' - P_1')dY$, is the marginal Y-supplier's surplus.

These are the exact same underlying changes; the transfer of dX from 2 to 1 and the transfer of dY from 1 to 2. But the MPKH reasoning now yields the reverse conclusions. The dY transfer accounts for all the increase in the size of the X pie so it can be recommended on "efficiency" grounds. The dX transfer merely redistributes the X pie so that transfer can be treated as a separate question of "equity."

One escapes numeraire illusion only by evaluating the transfers in terms of some third commodity Z not involved in the transfers. We use the array of prices in terms of Z assumed above.

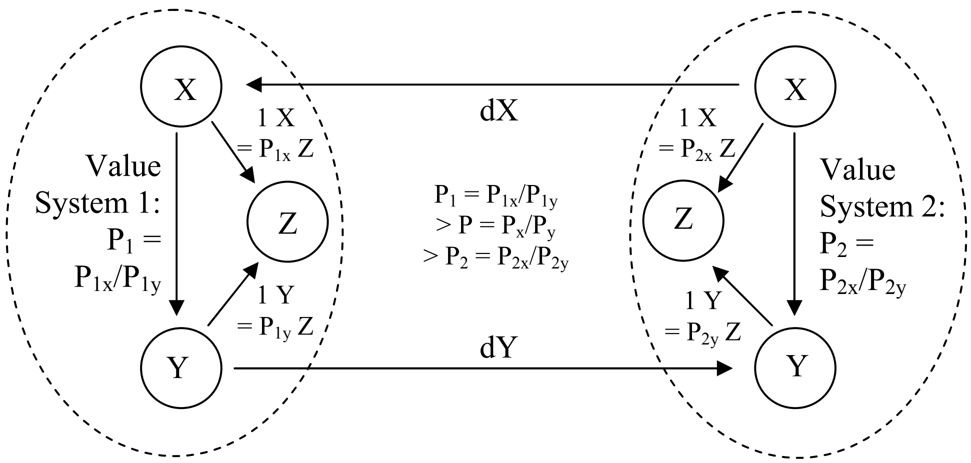

Figure 3. Two values for Y and for X in terms of Z

The exchange of dX and $dY = PdX$ is equal-valued at the P_x and P_y prices since $P_xdX = P_ydY$. But at the internal or "subjective" values in the two systems:

the change in Z value from the dX transfer is: $\Delta Z_x = (P_{1x} - P_{2x})dX > 0$, and
the change in Z value from the dY transfer is: $\Delta Z_y = (P_{2y} - P_{1y})dY > 0$.

The sum of the two increases is the total increase in the Z pie from the dX and dY transfers:

$$\Delta Z = \Delta Z_x + \Delta Z_y = (P_{1x} - P_{2x})dX + (P_{2y} - P_{1y})dY.$$

Since the exchange of dX and dY is made at the intermediate prices P_x and P_y where:

$$P_{1x} > P_x > P_{2x} \text{ and } P_{1y} < P_y < P_{2y}$$

we can compute the surpluses in each system in terms of Z. In system 1, the gain from receiving dX is $P_{1x}dX$ while the cost of losing dY is $P_{1y}dY$ so the net benefit to 1 is:

$$\begin{aligned}\Delta Z_1 &= P_{1x}dX - P_{1y}dY \\ &= (P_{1x} - P_x)dX + P_x dX - P_{1y}dY \\ &= (P_{1x} - P_x)dX + P_y dY - P_{1y}dY \\ &= (P_{1x} - P_x)dX + (P_y - P_{1y})dY > 0.\end{aligned}$$

Note that in the absence of numeraire illusion, the surplus to 1 now contains two terms, one term, $(P_{1x} - P_x)dX$, for what is demanded and one term, $(P_y - P_{1y})dY$, for what is supplied. In the customary treatment of consumer's and supplier's surplus, one of these terms drops out courtesy of numeraire illusion (see below).

Similarly the gain in system 2 from receiving dY and giving up dX is:

$$\Delta Z_2 = P_{2y}dY - P_{2x}dX = (P_{2y} - P_y)dY + (P_x - P_{2x})dX > 0$$

and the two benefits sum to the total Z benefit: $\Delta Z_1 + \Delta Z_2 = \Delta Z$.

There is no direct conversion of ΔZ into ΔX or ΔY since the Z must be converted into X or Y at the different rates internal to system 1 or system 2. For instance, $P_{1x}dX$ would be divided by P_{1y} to get the equivalent values in Y in system 1 while $P_{2x}dX$ would be divided by P_{2y} to get the equivalent Y value in system 2. Thus to arrive at ΔY, we would have to divide the different terms in the expression for ΔZ by the appropriate Y prices for each system:

$$\left(\frac{P_{1x}}{P_{1y}} - \frac{P_{2x}}{P_{2y}}\right)dX + \left(\frac{P_{2y}}{P_{2y}} - \frac{P_{1y}}{P_{1y}}\right)dY = (P_1 - P_2)dX + (1-1)dY = \Delta Y.$$

Note how that numeraire illusion appears in the mathematics as the zeroing out of the dY coefficient, i.e., $(1 - 1)$, in the calculation of ΔY. In a similar manner we could convert ΔZ into ΔX and the numeraire illusion would appear in the zeroing out of the dX coefficient in ΔX.

Perhaps it is convenient to use the MRS notation to summarize the three pies obtained from the three numeraires used to describe the same transaction of dX from 2 to 1 and dY from 1 to 2. The Z pie is:

$$\Delta Z = (MRS^1_{zx} - MRS^2_{zx})dX + (MRS^2_{zy} - MRS^1_{zy})dY.$$

Our main result about numeraire illusion is encapsulated in the fact that when one recalibrates ΔZ to a numeraire that is involved in the transfer (i.e., Y or X), then the coefficient for that term will drop out in that "size of the pie" formula. To get from ΔZ to ΔY, substitute Y for Z throughout the ΔZ formula, and similarly to get from ΔZ to ΔX:

$$\begin{aligned}\Delta Y &= (MRS^1_{yx} - MRS^2_{yx})dX + (MRS^2_{yy} - MRS^1_{yy})dY \\ \Delta X &= (MRS^1_{xx} - MRS^2_{xx})dX + (MRS^2_{xy} - MRS^1_{xy})dY.\end{aligned}$$

Numeraire illusion occurs since $MRS^i_{yy} = 1$ and similarly for X; the coefficient evaluating the transfer of the numeraire commodity in terms of the numeraire will always be $(1 - 1) = 0$. When numeraire illusion is avoided by evaluating the dX and dY changes in terms of some other non-involved commodity Z, then we see that *both* transfers add value:

$$\Delta Z_x = (MRS^1_{zx} - MRS^2_{zx})dX = (P_{1x} - P_{2x})dX > 0$$
$$\Delta Z_y = (MRS^2_{zy} - MRS^1_{zy})dY = (P_{2y} - P_{1y})dY > 0$$
$$\Delta Z = \Delta Z_x + \Delta Z_y.$$

The two parsings of ΔZ according to the effects of dX and dY and according to the two values systems can be given in tabular form.

Table 2. Effects of dX and dY transfers in terms of Z

	dX	**dY**	**Row Sum**
Value system 1	$P_{1x}dX = MRS^1_{zx}dX$	$-P_{1y}dY = -MRS^1_{zy}dY$	ΔZ_1
Value system 2	$-P_{2x}dX = -MRS^2_{zx}dX$	$P_{2y}dY = MRS^2_{zy}dY$	ΔZ_2
Column Sum	ΔZ_x	ΔZ_y	ΔZ

When the dX and dY transfers are evaluated in terms of the non-involved numeraire Z, the MPKH reasoning then gets no illusionary foothold to recommend either dX or dY by itself on "efficiency" grounds. This calculation also serves to emphasize that the usual judgment—that the market transfer of the commodity generates the consumer's and supplier's surpluses while the payment only serves to redistribute the surplus—is only a result of the numeraire illusion involved because the payment is made in the same commodity-metric as the surpluses are measured.

A Pollution Example From Law and Economics

The Chicago school of law and economics (social wealth maximization) applies the same logic of David Friedman's apple example quoted above to legal changes: "We now expand the analysis by applying Marshall's approach not to a transaction (John buys Mary's apple) but to a legal rule." [Friedman 2000, 20] Since so much of this approach to the economic analysis of law grew out of Ronald Coase's analysis of pollution [1960], such an example might be used to represent the methodology of the Chicago social wealth maximization school.

Take the first numeraire y to be money and take x to be the number of pollution permits.[13] Our points are independent of the question of polluter's rights or pollutee's rights, a question that has received much attention in the literature on Coase's Theorem. Hence we initially take a pollutee's rights perspective and then later take the opposite viewpoint. In our first example, person 1 is the polluter initially endowed with much money and few pollution rights while person 2 is the pollutee with the opposite relative endowments.

At the endowment point it might well be that $MRS^1_{yx} > MRS^2_{yx}$ so that there could some mutually voluntary exchanges of dy money for dx pollution permits between the polluter and

[13] See, for example, the SO_2 permits analyzed by my brother and his colleagues in Ellerman, Joskow, Schmalensee et al. 2000.

pollutee. So far so good; it is a Pareto improvement due to voluntary exchanges in the market for pollution permits with no need for the Kaldor-Hicks criterion or wealth maximization reasoning.

The problem comes when, say, a legal-economic analyst of the Chicago school uses the MPKH reasoning to analyze the transfer in pollution rights dx as an increase in "social wealth" while the payments dy are seen as a mere redistributive transfer with no effect on the size of social wealth (aside perhaps from minor income effects resulting from the intertwining of the size and distribution of the pie questions for finite changes). As social scientists, economists and economics-oriented legal theorists can recommend the efficiency change, the increase in social wealth due to the dx transfer. The advisability of the dy "redistribution" is, however, best left for philosophers, theologians, poets, environmentalists, and other non-economists (all as if the "efficiency" and "equity" parsing was an attribute of the underlying legal situation rather than just the choice of numeraire). Moreover the merely redistributive dy transfer might be plagued by various transaction costs that would actually reduce social wealth. Hence the most efficient outcome would be to make the social-wealth-increasing transfer dx to the polluter—in effect to switch that part of the endowment to the polluter—and avoid any of the deadweight social wealth losses due to the costs of the dy transaction. This would "mimic the market" in terms of increasing social wealth while avoiding the deadweight transaction costs.

All of these arguments and conclusions—representative of the Chicago school—change under the mere redescription of the situation by reversing the numeraires. Gains and losses are now to be expressed in the measuring rod of pollution rights ("x" in the example) and the transfers can be analyzed from the viewpoint of the new "social x pie." The money payment dy from the polluter to the pollutee increases "social wealth" (now measured in x) by $MRS^2_{xy}dy - MRS^1_{xy}dy$ while the dx transfer of pollution rights merely redistributes x with no effect on total social wealth as measured in x. One pollution permit is worth the same number of pollution permits to everyone: one. As professional scientists, economists can recommend the social-wealth-increasing transfer of the money dy from polluter to pollutee while the question of transferring the pollution rights dx is best left aside for philosophers and their ilk. There might even be some deadweight costs in social wealth associated with the transfer of the pollution rights dx so the most efficient outcome would then be to just reassign the money dy from the polluter to the pollutee. That would also "mimic the market" in terms of increasing social wealth while avoiding the deadweight transaction costs. [14]

The flaws in the MPKH reasoning have nothing to do with the Coase's Theorem controversy. The numeraire inversion analysis applies as in the above example if we start instead with the polluter's rights principle. Again y is money and x is pollution permits. But now take person 1 to be the pollutee relatively well-endowed with money but few pollution rights—the latter being endowed to person 2, the polluter. Again we might expect $MRS^1_{yx} > MRS^2_{yx}$ at the endowment point so that there could be some mutually beneficial voluntary exchange where the pollutee buys pollution rights dx from the polluter for the money dy. In the literature, this is sometimes viewed as the pollutee "bribing" the polluter to reduce pollution or it could be seen as the purchase of amenity rights to the public good of less pollution. While the endowment point

[14] This highlights the Pickwickian nature of the "mimic the market" rhetoric in Chicago-style law and economics that neglects the fact that market transactions involve actual payments, not "potential" ones.

might be controversial, there is little problem with the voluntary exchanges once the endowment is given.

The problem comes in the MPKH reasoning that would analyze the transfer of dx amenity rights from the polluter to the pollutee as an increase in the social y pie while the payment dy had zero effect on that pie. If there were deadweight transactions costs involved in the otherwise redistributive dy payments, then the most efficient outcome would be seen as the uncompensated transfer of the amenity rights dx from the polluter to the pollutee. Many environmentalists might applaud that outcome but one suspects that the disagreement is really with the polluter's rights endowment point. In any case, faulty economic reasoning should not be used to support the emotionally satisfying reassignment of the amenity rights dx from the polluter to the pollutee since that conclusion is easily reversed along with the numeraire.

With the amenity rights x taken as the numeraire, the payment dy from the pollutee to the polluter increases the size of the social pie (now measured in amenity rights) while the transfer of rights dx from polluter to pollutee is a "wash" since one pollution permit has the same value in terms of pollution permits to each party. And if there are transaction costs associated with transfer in amenity rights dx, then the MPKH-Chicago reasoning would conclude that the most efficient outcome is for the pollutee to make the bribe dy to the polluter but for the polluter to keep the pollution rights dx—a conclusion no doubt less congenial to environmentalists.

Consumer and Supplier Surpluses in a Competitive Market

The Normal Description

The main points in the critique of the MPKH reasoning should now be clear but it may be helpful to restate them in the familiar setting of the standard textbook rendition of the consumer and supplier surpluses in a competitive market for a commodity x. If the MPKH reasoning is sound, then surely it would work in the best possible case, a perfectly competitive market. If it fails in that ideal setting, then it could hardly fare better in the context of market distortions and failures where cost-benefit analysis and law-and-economics reasoning is applied.

The inverse redescription of the textbook competitive market model is also useful to help dispel the numeraire illusion that surpluses are associated with transfers in a non-numeraire good but not with transfers in the numeraire good. The inverted description can be used as an engine to discover non-invariance in conventional economic arguments.

There is a downward sloping demand curve; the quantity of x demanded x_d is a function $x_d = d(p)$ of the price in dollars per unit x. And there is an upward sloping supply curve; the quantity of x supplied is a function $x_s = s(p)$ of the price. Equilibrium occurs at a price p* at which the quantity demanded and supplied are equal: $x^* = d(p^*) = s(p^*)$.

Leaving aside the fine-grained controversy about measuring the consumer and supplier surpluses as not being germane to our analysis, the standard Marshallian definitions will be used. The total benefit to the consumer(s) in receiving x is measured in dollars by the area under the demand curve from 0 to x. If px was paid out to receive x, then the net gain or "consumer surplus" is the difference. In a similar manner, the area under the supply curve from 0 to x represents the loss measured in dollars to the supplier(s) in giving up x. If px was received in return for x, then the net gain or "supplier surplus" is the difference.[15]

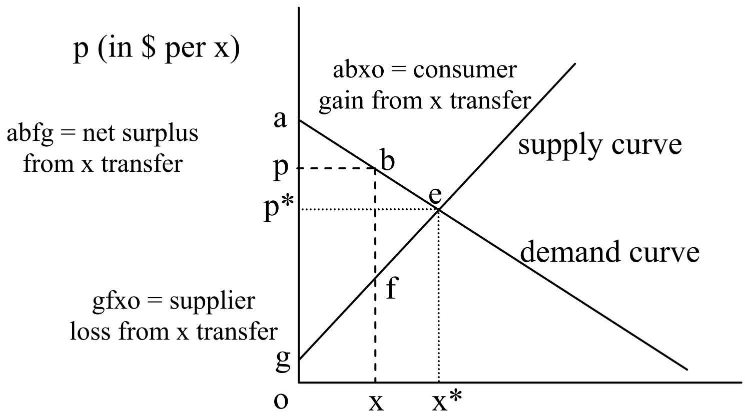

Figure 4. Standard supply and demand diagram

The most "efficient" amount of x to transfer is the x that maximizes the increase in the social $ pie (abfg) which is the equilibrium value x*. Many textbooks still use this MPKH reasoning to "explain" the "efficiency" of the competitive equilibrium (in this market, the exchange of x* in return for p*x* dollars). The Paretian explanation (using up all the opportunities for mutually beneficial exchange) is usually also given as if the two accounts were equivalent.

But the difference between the two accounts becomes clear as soon as we take the MPKH reasoning seriously enough to ask about the efficiency role of the p*x* payment. From the Paretian viewpoint, it is necessary to make the exchange *mutually* beneficial, a Pareto improvement, so the x* transfer without the p*x* transfer does not pass Paretian muster. But from the KH-efficiency point of view, the payment p*x* is redistributive; it does not change the size of the social $ pie. Thus in the MPKH account, the p*x* transfer is only a question of "equity," not "efficiency."

In spite of the MPKH reasoning being developed to facilitate economics giving "professional" or "scientific" advice to public policy, the reasoning is in fact so dubious that it reverses itself after a mere redescription of the same market with reversed numeraires. We turn now to that inverse description of the same market.

The Inverse Description

The supply curve provides the functional relationship giving the amount of x that is supplied if the revenue $R = s(p)p$ is paid for it. We might think of the x-supplier as the R-demander and the inverse or reciprocal $p' = 1/p$ as the unit price of a dollar-spent-on-x[16] in terms of x (where we may assume $0 < p < \infty$ and thus p' is in the same range). Thus the revenue demanded as a

[15] For simplicity, we might assume just one consumer and one supplier, and thus the apostrophe is before the "s" in consumer's and supplier's surpluses.

[16] Intuitively, the commodity "dollars-spent-on-x" could be thought of as money earmarked in a budget to be spent on x. The amount of this commodity supplied to or demanded from the market will depend on its price $p' = 1/p$ in terms of the numeraire x. Like an earmarked budget item, R units of this commodity can only be exchanged for p'R units of x.

function of p' is $R_d(p') = R(1/p') = s(1/p')/p'$. This is the revenue (money-spent-on-x) demand function in the redescribed market interpreting x-supply as R-demand:

$$\text{Revenue demand curve: } R_d(p') = s(1/p')/p'.$$

For the illustrative case of a linear supply curve $x_s = cp - d$ (with c and d non-negative), the revenue demand curve is the downward-sloping curve [in the positive (R,p') quadrant]:

$$\text{Revenue demand curve for a linear supply: } R_d(p') = \frac{c}{p'^2} - \frac{d}{p'}.$$

The x demand curve gives the functional relationship between the amount of x that is demanded and the money or revenue R = d(p)p supplied for it. We might think of the x-demander as a money-spent-on-x or revenue-supplier. The revenue supplied as a function of its unit price p' is thus:

$$\text{Revenue supply curve: } R_s(p') = d(1/p')/p'.$$

Since the revenue R(p) = d(p)p is the product of a decreasing and an increasing function of p, it is not necessarily monotonic and the revenue supply curve might be backward bending [in the positive (R,p') quadrant]. In the illustrative case of a linear demand curve $x_d = -ap + b$ (with a and b non-negative), the revenue supply curve is indeed backward bending:

$$\text{Revenue supplied for a linear demand curve: } R_s(p') = \frac{b}{p'} - \frac{a}{p'^2}.$$

An illustrative redescription of the x & R market is given in the following diagram.

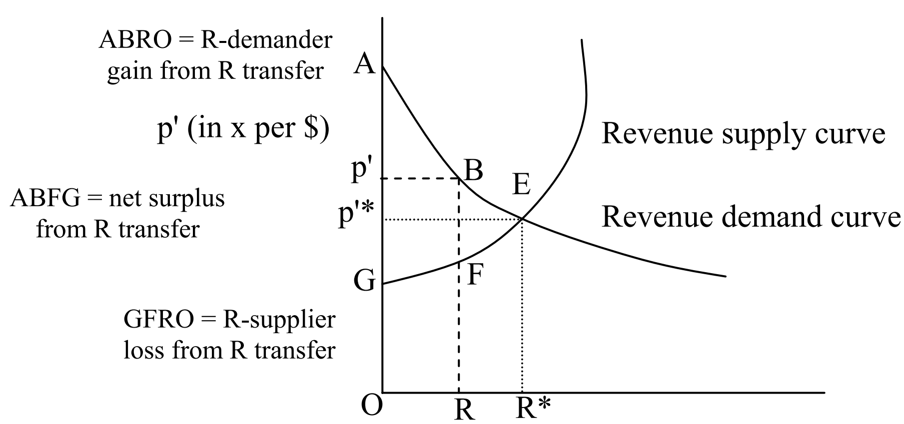

Figure 5. Inverse description of market as supply and demand for R using x as numeraire

The equilibrium price p'* in the redescribed market is the p' where

$$R_s(p') = d(1/p')/p' = s(1/p')/p' = R_d(p').$$

Multiplying through by p', equilibrium occurs at the p' where d(1/p') = d(p) = x_d = x_s = s(p) = s(1/p') which are the original equilibrium conditions. The quantity x demanded and supplied is equal at the price p*, the equilibrium price in the market for x, so p'* = 1/p*. At p'*, the equilibrium amount of the revenue R* is R_s(p'*) = d(1/p'*)/p'* = d(p*)p* = x*p*. The amount of x paid for R* is the price times the quantity: p'*R* = x*p*/p* = x*. Thus the redescribed market gives exactly the same equilibrium—just looked at in the inverted way as the market for the supply and demand for money-spent-on-x with the payments made in quantities of x. However, the constructs of the MPKH methodology change completely with the change in numeraire.

The area under the revenue demand curve (ABRO in the above diagram) from 0 to R gives the total gain to the $-demander (the x-supplier), expressed in the numeraire x, from receiving R. The area under the revenue supply curve from 0 to R (GFRO) gives the total loss to the $-supplier (the x-demander) from giving up R. The difference (ABFG) gives the "total social surplus," the increase in the "social x pie," from the R transfer from the $-supplier to the $-demander. The transfer of the payment for R, p'R = (1/p)px = x in the opposite direction is a mere redistribution of x that does not change the size of the social x pie.

The most efficient transfer of R is the amount that maximizes the increase in the social x pie—which is R*. That transfer of R* (= p*x*) from the $-supplier (x-demander) to the $-demander (x-supplier) is the "efficiency" part of the transaction. The transfer of payment p'*R* = (1/p*)p*x* = x* from the $-demander to the $-supplier does not affect the size of the social x pie so it is the "equity" part of the total change—all according to the MPKH reasoning.

Thus in the generic textbook supply and demand competitive model, we see the ordinary description of the model with money as the numeraire and we see the inverse description with the commodity x as the numeraire. The underlying properties of the model (e.g., equilibrium value of x*, equilibrium price ratio of $ per x of p*, and the equilibrium amount of money R* transacted) are all the same under the redescription. But the MPKH parsing of the "efficiency" part and the "equity" part of the total change reverse themselves under the redescription as indicated in the following table.

Table 3. Efficiency-Equity Reversal in Standard Competitive Market Analysis

	Normal Description (Money = numeraire)	**Inverse Description (x = numeraire)**
Transfer that increased size of "social pie"	Commodity x* from x-supplier to x-demander.	Commodity R* from R-supplier (i.e., x-demander) to R-demander (x-supplier).
Transfer that only redistributed "social pie"	Payment R* = p*x* from x-demander to x-supplier.	Payment x* = p'*R* from R-demander (i.e., x-supplier) to R-supplier (x-demander).

Since the "efficiency" (increased size of "social pie") analysis reverses itself under the mere numeraire reversal, the standard conclusion of MPKH reasoning—that the proposed project (x* in the example) can be recommended on efficiency grounds—cannot be sustained.

Final remarks

Numeraire illusion is a mathematical fact (like $MRS_{yy} = 1$). The question is: Where is economic reasoning mislead by numeraire illusion into making non-invariant conclusions? There is a whole research programme to conduct an intellectual audit across economics to see where numeraire illusion might have lead to error as it did in the Marshall-Pigou-Kaldor-Hicks tradition of welfare economics. Numeraire inversion is an engine of discovery for that research programme.

Our focus here has been on cost-benefit analysis, Chicago-style (wealth maximization) law and economics, and other areas of applied welfare economics based on the MPKH methodology. The common pattern is that the benefits and costs of a proposed change or project are evaluated using the measuring rod of money as the numeraire. If the net benefits are positive, then in theory the winners could compensate the losers to make the overall change a Pareto improvement. Then the MPKH reasoning is used to represent the project by itself as (courtesy of numeraire illusion) the only increase in the social pie measured by the money metric and thus as something that can be recommended by economists on efficiency grounds—even though the project sans compensation is only a Kaldor-Hicks improvement, not a Pareto improvement. The compensation is, for the same illusionary reason, represented as only a redistribution of the social pie, a question of equity, not efficiency.

> The purpose of considering hypothetical redistributions is to try and separate the *efficiency* and *equity* aspects of the policy change under consideration. It is argued that whether or not the redistribution is actually carried out is an important but *separate* decision. The mere fact that is it possible to create potential Pareto improving redistribution possibilities is enough to rank one state above another on efficiency grounds. [Boadway and Bruce 1984, 97]

Richard Posner makes a similar point in the context of Chicago-style law and economics. He notes that "Kaldor-Hicks efficiency" leaves distributive considerations to one side.

> But to the extent that distributive justice can be shown to be the proper business of some other branch of government or policy instrument…, it is possible to set distributive considerations to one side and use the Kaldor-Hicks approach with a good conscience. This assumes, …, that efficiency in the Kaldor-Hicks sense—making the pie larger without worrying about how the relative size of the slices changes—is a social value. [Posner 2000, 1154-5]

This pattern of reasoning—which assumes that the parsing of a proposed change into "efficiency" (change in size of pie) and "equity" (change in shares in pie) parts is a description-invariant property of the change—runs the length and breadth of the conventional law and economics literature and it is the warhorse of cost-benefit analysis including project evaluation and other parts of applied welfare economics.

As is clear from the numeraire reversals, there are simply no economic grounds to declare the "project" as an "increase in efficiency" and the "compensation" as a "mere redistribution" rather than exactly the reverse. Both the "project" and the "compensation" are reallocations of resources that will each benefit some people and hurt others. The efficiency-equity analysis of the Marshall-Pigou-Kaldor-Hicks tradition does not provide solid economic grounds to claim that either partial change can be recommended by itself on "efficiency" grounds. [17]

Appendix: Vector Maximization with Simple and Complex Changes

Simple Changes as Project and Compensation

First Description

Consider the vector maximization problem:

$$[f(x,y), g(x,y)]$$

where both functions are differentiable with non-zero partial derivatives. As in the schematic argument given above, we could take the first function as a value function representing the winner in the "project" dx while the second function represents the loser in the "project" which would mean being the winner in the "compensation." Since our purpose is to illustrate numeraire illusion in the first description taking one involved variable as the numeraire and then again in the reversed or inverted description taking the other involved variable as numeraire, we drop the non-involved variable z.

The variables x and y now have the same value in both value functions so that they could represent either private goods or public goods. For instance, if x is a private good and if $x = x_1$ represents the amount consumed by the first party, then the amount consumed by the second party would be determined by some function $x_2 = h(x)$. If there was a constant total x^0, then $x_2 = x^0 - x$. Thus if g was a utility function, then $x = x_1$ might occur in it as $g(x^0 - x, y)$. Alternatively, x or y could be a public good. In either case, there is no assumption that the partial derivatives are positive, only non-zero.

We assume that the initial point (x,y) is not a vector maximum so there exists a marginal change (dx,dy) away from the initial point that will increase both functions, i.e., that make a vector improvement:

$$df = f_x dx + f_y dy > 0$$
$$dg = g_x dx + g_y dy > 0.$$

Moreover we may assume that dx by itself will increase f but decrease g while dy will increase g but decrease f:

$$f_x dx > 0 > f_y dy$$
$$g_x dx < 0 < g_y dy$$

[17] While beyond the scope of this paper, it might be noted that this conclusion is congruent with the Wicksell-Buchanan perspective in political economy [Buchanan 1999]. Instead of using MPKH reasoning to supply an "efficiency" gloss to "the planner" [Boadway and Bruce 1984, 9], it is the job of democratic politics to work out changes that are mutually voluntary on the part of all those whose rights are affected.

so that neither dx nor dy by itself would constitute a vector improvement. If we think of dx as the "project" and dy as the "compensation," then f is the beneficiary of the project while g is the loser but both improve when both the project dx and the compensation dy are undertaken.

We start with the "normal description" where y is taken as the numeraire. However, it is more convenient to take the given dy as the unit instead of y so let "t" be the number of units of dy [note that dy could be positive or negative]. The benefit and cost of the project dx is evaluated in terms of t. The project dx generates the gross benefit of $f_x dx$ in terms of f but we can re-express that in terms of the equivalent number of units of t where "equivalent" means leaving f constant. One unit of t, i.e., dy, gives the reduction $f_y dy$ in f so let Δt_f be the units of t that would just leave f the same when dx is undertaken, i.e.,

$$f_x dx + \Delta t_f f_y dy = 0$$

so that we have: $$\Delta t_f = -f_x dx / f_y dy > 0.$$

Since $f_x dx + f_y dy > 0 = f_x dx + \Delta t_f f_y dy$ and $0 > f_y dy$, we have that:

$$1 < \Delta t_f.$$

An equivalent approach would be to define the marginal rate of substitution of t per x as:

$$MRS^f_{tx} = f_x / (-f_y dy)$$

so that the change in t, Δt_f, that would leave f constant after implementing dx is:

$$\Delta t_f = MRS^f_{tx} dx = -f_x dx / f_y dy$$

which is the same solution.[18]

We could similarly define the change in t, Δt_g, that would hold g constant when dx is implemented by:

$$g_x dx + \Delta t_g g_y dy = 0 \text{ or } \Delta t_g = -g_x dx / g_y dy > 0.$$

Since $g_x dx + g_y dy > 0 = g_x dx + \Delta t_g g_y dy$ and $0 < g_y dy$ we have that:

$$1 > \Delta t_g.$$

Using the MRS approach, we could define $MRS^g_{tx} = -g_x / g_y dy$ so that the change in t that would leave g constant when dx is implemented would be the same:

$$MRS^g_{tx} dx = -g_x dx / g_y dy = \Delta t_g.$$

[18] Since $f_x dx > 0 > f_y dy$, the minus sign is inserted in the MRS to insure that the result is positive. Similarly for the other MRSs considered below.

Since $f_y dy < 0$, the change dy decreases f and thus the number of units of dy needed to just counterbalance the benefit of the project dx, namely Δt_f, is the t-measure of the benefit of dx. But the project dx also has costs in that it affects g. The t-measure of that cost is the number of units of dy that would just leave g the same, namely Δt_g. Hence the net benefit Δt of the project dx measured in t is:

$$\Delta t = \Delta t_f - \Delta t_g > 0.$$

That is the increase in the size of the "t pie" resulting from the project dx.

Now we can compute the net benefit of implementing both the project dx and the compensation together. Using the MRS approach where $MRS^f_{ty} = f_y/(-f_y dy)$, the benefit of implementing both dx and dy is:

$$MRS^f_{tx}\, dx + MRS^f_{ty}\, dy = -f_x dx/f_y dy - f_y dy/f_y dy = \Delta t_f - 1.$$

Similarly with $MRS^g_{ty} = -g_y/g_y dy$, the cost of implementing both dx and dy is:

$$MRS^g_{tx}\, dx + MRS^g_{ty}\, dy = -g_x dx/g_y dy - g_y dy/g_y dy = \Delta t_g - 1.$$

Here is the numeraire illusion. What the dy compensation subtracts from the project beneficiary f, it also subtracts from the costs to the project loser g, so the net benefit of implementing both dx and dy measured in t is the same:

$$(\Delta t_f - 1) - (\Delta t_g - 1) = \Delta t_f - \Delta t_g + (1 - 1) = \Delta t.$$

Thus it is said in cost-benefit (CB) analysis that the "compensation" dy does not change the size of the t pie.[19] The total change is parsed into the "efficiency" part dx which increases the size of the pie and the "equity" part dy which does not change the size of the pie. On efficiency grounds, the project dx is recommended in CB analysis while the compensation dy can be treated separately as a question of equity.

While the dy change does not alter the size of the t pie, it does change the distribution of the increases between the f and g functions. The increase Δt in the size of the t pie is the net benefit to be distributed between the functions. If only the minimum compensation of Δt_g is paid, then g stays constant and all the increase would accrue to f. If the maximum compensation of Δt_f were paid then f stays constant and all the increase goes to g. And if the given dy is the compensation actually made, then it corresponds to $t = 1$ and $\Delta t_f > 1 > \Delta t_g$ so f would increase by $\Delta t_f - 1$ and g would increase by $1 - \Delta t_g$ with both increases measured in t and sum to Δt.

As a check, we might consider that the rate of change of f with respect to t is

[19] "It should be emphasized that pure transfers of purchasing power from one household or firm to another per se should be typically attributed no value." [Boadway 2000, 30] Or again, "pure transfers of funds among households, firms and governments should themselves have no effect on project benefits and costs." [Boadway 2000, 35]

$$\frac{\partial f}{\partial t} = \frac{\partial f(x, y + tdy)}{\partial t} = f_y dy < 0$$

so to convert the increase $df = f_x dx + f_y dy > 0$ into the equivalent amount of t, divide the increase df by $-f_y dy > 0$ to obtain:

$$-df/f_y dy = -(f_x dx + f_y dy)/f_y dy = \Delta t_f - 1 > 0.$$

The rate of change of g with respect to t is:

$$\frac{\partial g}{\partial t} = \frac{\partial g(x, y + tdy)}{\partial t} = g_y dy > 0$$

so to convert the increase $dg = g_x dx + g_y dy > 0$ into the equivalent amount of t, we will divide by $g_y dy > 0$ to obtain:

$$dg/g_y dy = (g_x dx + g_y dy)/g_y dy = 1 - \Delta t_g > 0.$$

Hence the net benefit is split into the two parts that accrue to f and g when both dx and dy are undertaken:

$$\Delta t = (\Delta t_f - 1) + (1 - \Delta t_g).$$

Intuitively, we might think of t as the number of units dy of the numeraire which subtract from f and add to g. In this "normal description," the cost-benefit analysis of the "project" dx and the "compensation" dy was carried out in terms of the numeraire t. The same changes can be analyzed in the "inverted description" using the numeraire s, which is x measured in the units dx (i.e., s = 1 corresponds to dx units of x). Then we can reverse the role of dy and dx and get similar results.

The flaws in the cost-benefit analysis lie not in the mathematics of either case but in the reasoning that the changes involving the numeraire do not affect the size of the pie ("numeraire illusion") while the non-numeraire changes yield an "efficient" increase in the size of the pie. In the inverted description, we will see that the same reasoning gives the reverse conclusions, i.e., that dy should be undertaken as a matter of "efficiency" while dx is set aside as a question of "equity." The efficiency-equity parsing is only an attribute of the description in terms of one numeraire or the other, not an attribute of the underlying situation. Since any sound policy recommendations should at least survive a mere redescription of the same situation using a different numeraire, the "scientific" recommendation of the efficiency part cannot be sustained. We turn now to the inverted description.

Inverted Description

In the inverted description, x is taken as the numeraire but dx is taken as the unit and the scale factor s is the number of units of dx. The "project" is now taken as dy with dx treated as the "compensation" where the combined effect, as before, increases both f and g. When the project dy is undertaken, then let Δs_f be the number of units of s that would leave f constant so:

$$\Delta s_f\, f_x dx + f_y dy = 0 \text{ or } \Delta s_f = -f_y dy/f_x dx = 1/\Delta t_f > 0.$$

Note that under numeraire inversion, the amount Δs_f (in terms of s) that holds f constant is the multiplicative inverse of the amount Δt_f (in terms of t) that holds f constant.

Since $f_x dx + f_y dy > 0 = \Delta s_f\, f_x dx + f_y dy$ and $f_x dx > 0$, we have

$$1 > \Delta s_f > 0.$$

Thinking in terms of marginal rates of substitution, we could define

$$MRS^f_{sy} = -f_y/f_x dx$$

so the change in s that would hold f constant when dy is undertaken is again:

$$MRS^f_{sy} dy = -f_y dy/f_x dx = \Delta s_f > 0.$$

The change in s that would hold g constant when dy is undertaken is:

$$\Delta s_g\, g_x dx + g_y dy = 0 \text{ or } \Delta s_g = -g_y dy/g_x dx = 1/\Delta t_g > 0.$$

Note that under numeraire inversion, the amount Δs_g (in terms of s) that holds g constant is the multiplicative inverse of the amount Δt_g (in terms of t) that holds g constant.

Using the MRS approach, $MRS^g_{sy} = -g_y/g_x dx$ so that the change in s that would leave g constant when dy is implemented would be the same:

$$MRS^g_{sy} dy = -g_y dy/g_x dx = \Delta s_g.$$

Since $g_x dx + g_y dy > 0 = \Delta s_g\, g_x dx + g_y dy$ and $0 > g_x dx$ we have that:

$$1 < \Delta s_g.$$

When dy is undertaken, g increases and the increase in g, measured in terms of the offsetting change in s, is Δs_g. Yet dy has a cost since it reduces f so the offsetting change in s to hold f constant is Δs_f. Hence the net benefit from dy measured in terms of s is the increase Δs in the size of the s pie:

$$\Delta s = \Delta s_g - \Delta s_f > 0.$$

If the "compensation" dx is also undertaken, then the change in g measured in terms of s (remembering that dx corresponds to s = 1) is: $\Delta s_g - 1$. The change in f measured in terms of s is then: $\Delta s_f - 1$, so the net benefit of both the project dy and the compensation dx measured in s is:

$$(\Delta s_g - 1) - (\Delta s_f - 1) = \Delta s_g - \Delta s_f \ + (1 - 1) = \Delta s.$$

Hence it is said that the compensation dx does not change the size of the s pie.

But dx does change the distribution of the benefits between f and g. When the project dy is undertaken and only the cost Δs_f is paid, then all the benefit accrues to g. If dy is undertaken and the maximum cost of Δs_g is paid then all the benefit accrues to f. But when the project dy and the compensation dx (s = 1) are both undertaken, then the increase in g measured in s is $\Delta s_g - 1$ while the increase in f measured in s is $1 - \Delta s_f$, and the net benefit is the sum of the two parts:

$$\Delta s = (\Delta s_g - 1) + (1 - \Delta s_f).$$

As a check, we might consider that the rate of change of f with respect to s is

$$\frac{\partial f}{\partial s} = \frac{\partial f(x + sdx, y)}{\partial s} = f_x dx > 0$$

so to convert the increase $df = f_x dx + f_y dy > 0$ into the equivalent amount of s, divide the increase df by $f_x dx > 0$ to obtain:

$$df/f_x dx = (f_x dx + f_y dy)/f_x dx = \ 1 - \ \Delta s_f > 0.$$

The rate of change of g with respect to s is:

$$\frac{\partial g}{\partial s} = \frac{\partial g(x + sdx, y)}{\partial s} = g_x dx < 0$$

so to convert the increase $dg = g_x dx + g_y dy > 0$ into the equivalent amount of s, we will divide dg by $-g_x dx > 0$ to obtain:

$$-dg/g_x dx = -(g_x dx + g_y dy)/g_x dx = \Delta s_g - 1 > 0.$$

Hence the net benefit is split into the two parts that accrue to f and g when both dx and dy are undertaken:

$$\Delta s = (\Delta s_g - 1) + (1 - \Delta s_f).$$

Intuitively, we might think of s as the number of units ds of the numeraire which subtract from g and add to f. The compensation dx corresponds to s = 1. The project dy and the compensation dx leave both parties better off. But the project dy alone accounts for the increase Δs in the size of the s pie while the compensation dx only redistributes the Δs benefits between g and f. Hence applying the usual efficiency-equity reasoning of cost-benefit analysis, we could recommend the project dy on efficiency grounds while leaving the compensation dx aside as a separate question of equity. Yet these are the same changes dx and dy which under the first description lead to the "efficiency" recommendation that dx be undertaken while dy was then a question of "equity."

Since the efficiency recommendation reverses itself under the redescription of the same situation with an inverted numeraire, that recommendation cannot be sustained.

The change dx is worth more to f than dy while the dy change is worth more to g than dx so both functions will increase when both dx and dy are made. There is no reason to recommend solely dx or dy on "efficiency" grounds. But when the changes dx and dy are evaluated not in terms of f or g but in terms of either x (measured in s units of dx) or y (measured in t units of dy), then there is the numeraire illusion of the change in the numeraire appearing to have no net benefit. In each case, the non-numeraire change appeared to yield all the net benefit while the changes involving the numeraire variable seemed to have only a redistributive effect (numeraire illusion). The recommendation that the non-numeraire change be made on "efficiency" grounds is only based on the illusion created by the choice of numeraire. Reverse the numeraires and the "efficiency" recommendation is reversed too.

This treatment with simple scalar changes dx and dy will now be directly generalized to complex changes represented by vectors.

Complex Changes as Project and Compensation

First Description

Consider the vector maximization problem:

$$[f(x_1,\ldots,x_m,y_1,\ldots,y_n),\ g(x_1,\ldots,x_m,\ y_1,\ldots,y_n)].$$

Writing $X = (x_1,\ldots,x_m)$ and $Y = (y_1,\ldots,y_n)$, the vector of functions could be written as:

$$[f(X,Y),\ g(X,Y)].$$

At the initial point (X,Y), the gradients of partial derivatives are

$$\nabla f(X,Y) = \left(\partial f/\partial x_1,\ldots\partial f/\partial x_m, \partial f/\partial y_1,\ldots,\partial f/\partial y_n\right)$$
$$\nabla g(X,Y) = \left(\partial g/\partial x_1,\ldots\partial g/\partial x_m, \partial g/\partial y_1,\ldots,\partial g/\partial y_n\right).$$

Assume that the initial point is not a vector maximum so we may assume there is a marginal variation $(dx_1,\ldots,dx_m,dy_1,\ldots,dy_n) = (dX,dY)$, such that

$$df = \nabla f(X,Y)(dX,dY) > 0$$
$$dg = \nabla g(X,Y)(dX,dY) > 0.$$

We further assume that dX by itself would increase f but would decrease g while dY has the opposite effect:

$$\nabla f(X,Y)(dX,0) > 0 > \nabla f(X,Y)(0,dY)$$
$$\nabla g(X,Y)(dX,0) < 0 < \nabla g(X,Y)(0,dY).$$

Note that the "0" substituted for dX or dY would be the zero vector with the appropriate number of components.

In this first description, we take dX as the "project" and dY as the "compensation" where both functions are increased if both the project and compensation are implemented. Since both the project and compensation are vectors, we let s be the scalar scale factor for the project dX so that s = 1 corresponds to sdX = dX, and let t be the scale factor for the compensation dY. In this description, the scale factor t for the compensation vector dY will be taken as the numeraire. Since the compensation is a vector dY, t functions as the scale factor for the "composite commodity" numeraire of dY.

Since the project dX has a positive effect on f and the compensation a negative effect, the scale change Δt_f in the compensation that would result in no change in f would satisfy:

$$\nabla f(X,Y)(dX,0) + \Delta t_f \nabla f(X,Y)(0,dY) = 0$$

so that:

$$\Delta t_f = -\nabla f(X,Y)(dX,0/\nabla f(X,Y)(0,dY).$$

Since $df = \nabla f(X,Y)(dX,0) + \nabla f(X,Y)(0,dY) > 0$ we have

$$\Delta t_f \nabla f(X,Y)(0,dY) < \nabla f(X,Y)(0,dY)$$

and thus recalling that $\nabla f(X,Y)(0,dY) < 0$, we have

$$\Delta t_f > 1.$$

Intuitively the idea was to evaluate the benefit to f of the project dX in terms of the increase in the scale factor t that would just leave f constant when dX is implemented.

This can also be explained in terms of marginal rates of substitution. The marginal rate of substitution of t per any variable (call it "x") would be $MRS^f_{tx} = -f_x/\nabla f(X,Y)(0,dY)$ so for all the variables, it would be the vector[20]

$$-\nabla f(X,Y)/\nabla f(X,Y)(0,dY).$$

Hence the desired amount of change in t that would leave f unchanged when dX is undertaken is the scalar product of dX with that MRS:

$$\Delta t_f = -\nabla f(X,Y)(dX,0)/\nabla f(X,Y)(0,dY)$$

which gives the same solution.

We may similarly define the change in scale t necessary to hold g constant as:

[20] The minus sign is inserted since $\nabla f(X,Y)(0,dY) < 0$ and we want the overall change in t to measure the increase rather than reduction in t. Similarly for the other MRSs defined below.

$$\nabla g(X,Y)(dX,0) + \Delta t_g \nabla g(X,Y)(0,dY) = 0$$

which solves to: $\Delta t_g = -\nabla g(X,Y)(dX,0)/\nabla g(X,Y)(0,dY)$.

Since $\nabla g(X,Y)(dX,0) + \nabla g(X,Y)(0,dY) > 0$ and $\nabla g(X,Y)(0,dY) > 0$, we have that

$$1 > \Delta t_g.$$

This could also be motivated by considering the marginal rate of substitution of t per any x $MRS^g_{tx} = -g_x/\nabla g(X,Y)(0,dY)$ and thus the vector MRS would be $-\nabla g(X,Y)/\nabla g(X,Y)(0,dY)$. Hence the scale change Δt_g that would hold g constant when dX is implemented is the scalar product of dX times that vectorial MRS:

$$\Delta t_g = -\nabla g(X,Y)(dX,0)/\nabla g(X,Y)(0,dY)$$

which gives the same solution.

Thus we have that the benefit of dX measured in terms of t is Δt_f (larger than 1) while the cost measured in terms of t is Δt_g (smaller than 1) so the net benefit Δt is:

$$\Delta t = \Delta t_f - \Delta t_g > 0.$$

That is the increase in the size of the "t pie" resulting from the project dX. Intuitively, when dX is implemented, then Δt_f is the benefit to f measured in terms of t. But the project dX also has a cost in that it reduces g so the cost Δt_g is the amount that would improve g just enough to keep it constant when dX is implemented. The t-benefit minus the t-cost is the net t-benefit Δt.

Now we can compute the t-benefit and t-cost of implementing both the project dX and the compensation dY. To compute the t-benefit, multiply the vectorial MRS for f by the total change (dX,dY) to get:

$$-\nabla f(X,Y)(dX,dY)/\nabla f(X,Y)(0,dY) = \Delta t_f - 1.$$

To compute the t-cost, multiply the vectorial MRS for g by the total change (dX,dY) to get:

$$-\nabla g(X,Y)(dX,dY)/\nabla g(X,Y)(0,dY) = \Delta t_g - 1$$

so the net benefit of implementing both the project and the compensation is:

$$(\Delta t_f - 1) - (\Delta t_g - 1) = \Delta t_f - \Delta t_g + (1 - 1) = \Delta t.$$

Thus it is said in cost-benefit analysis that paying the compensation dY does not change the total benefit of the project dX.

While the dY change does not alter the size of the t pie, it does change the distribution of the increases between the f and g functions. The increase Δt in the size of the t pie is the net benefit to be distributed between the functions. If only the minimum compensation of Δt_g is paid, then g stays constant and all the increase would accrue to f. If the maximum compensation of Δt_f were paid then f stays constant and all the increase goes to g. And if the given dY is the compensation actually made, then it corresponds to t = 1 and $\Delta t_f > 1 > \Delta t_g$ so f would increase by $\Delta t_f - 1$ and g would increase by $1 - \Delta t_g$ where both increases are measured in t and where they sum to Δt.

As a check, we might consider that the rate of change of f with respect to t is

$$\frac{\partial f}{\partial t} = \frac{\partial f(X, y_1 + tdy_1, ..., y_n + tdy_n)}{\partial t} = \nabla f(X, Y)(0, dY) < 0$$

so to convert the increase df > 0 into the equivalent amount of t, divide the increase df by the positive $-\partial f/\partial t$ to obtain the positive df expressed in t units as the positive quantity:

$$-\nabla f(X,Y)(dX,dY)/\nabla f(X,Y)(0,dY) = \Delta t_f - 1 > 0.$$

The rate of change of g with respect to t is:

$$\frac{\partial g}{\partial t} = \frac{\partial g(X, y_1 + tdy_1, ..., y_n + tdy_n)}{\partial t} = \nabla g(X, Y)(0, dY) > 0$$

so to convert the increase dg > 0 into the equivalent amount of t, we will divide by $\partial g/\partial t$ to obtain:

$$\nabla g(X,Y)(dX,dY)/\nabla g(X,Y)(0,dY) = 1 - \Delta t_g > 0.$$

Hence the net benefit is split into the two positive parts that accrue to f and g when both dx and dy are undertaken:

$$\Delta t = (\Delta t_f - 1) + (1 - \Delta t_g).$$

That corresponds to the normal description with the benefits and costs measured in terms of the numeraire t, the scale factor for the composite bundle dY. CB analysis would parse the total change (dX,dY) into the "efficiency" part dX which increases the size of the pie and the "equity" part dY which does not change the size of the pie. On efficiency grounds, the project dX would be recommended while the compensation dY can be treated separately as a question of equity.

Inverted Description

Now we give the inverted description of the same total change (dX,dY) where dY is taken as the project, dX is the compensation, and the numeraire is the scale factor s for the composite commodity dX which functions as the compensation. The change Δs_f in the scale factor s that would leave f unchanged when the project dY is implemented satisfies:

$$\Delta s_f \nabla f(X,Y)(dX,0) + \nabla f(X,Y)(0,dY) = 0$$

which solves to:

$$\Delta s_f = -\nabla f(X,Y)(0,dY)/\nabla f(X,Y)(dX,0) = 1/\Delta t_f > 0.$$

Intuitively, the marginal rate of substitution of s per any variable (call it "y") would be $MRS^f_{sy} = -f_y/\nabla f(X,Y)(dX,0)$. Hence for all the variables, it would be the vector $-\nabla f(X,Y)/\nabla f(X,Y)(dX,0)$. Hence the amount of change in s that would leave f unchanged when dY is undertaken is:

$$\Delta s_f = -\nabla f(X,Y)(0,dY)/\nabla f(X,Y)(dX,0)$$

which is the same solution. Since $\nabla f(X,Y)(dX,0) + \nabla f(X,Y)(0,dY) > 0$, we have

$$1 > \Delta s_f > 0.$$

Since the project dY has a negative impact on f, Δs_f is the scale of the compensation dX that would need to be paid to hold f constant. It is the s-cost of the project dY.

The project dY improves g and the measure Δs_g of that benefit is the increase in s that would just counterbalance the increase in g, i.e.,

$$\Delta s_g \nabla g(X,Y)(dX,0) + \nabla g(X,Y)(0,dY) = 0$$

which solves to: $\Delta s_g = -\nabla g(X,Y)(0,dY)/\nabla g(X,Y)(dX,0) = 1/\Delta t_g$.

Since $\nabla g(X,Y)(dX,0) + \nabla g(X,Y)(0,dY) > 0$ and $\nabla g(X,Y)(dX,0) < 0$, we have that:

$$\Delta s_g > 1$$

which we also know since $\Delta s_g = 1/\Delta t_g$ and $1 > \Delta t_g > 0$.

This could also be motivated by considering the marginal rate of substitution of s per any y $MRS^g_{sy} = -g_y/\nabla g(X,Y)(dX,0)$ and thus the vector MRS would be $-\nabla g(X,Y)/\nabla g(X,Y)(dX,0)$. Hence the scale change Δs_g of dX that would just counteract the increase in g due to dY is the scalar product of dY times that vectorial MRS:

$$\Delta s_g = -\nabla g(X,Y)(0,dY)/\nabla g(X,Y)(dX,0)$$

which gives the same solution.

When project dY is implemented, then the scale increase Δs_g is the s measure of the benefit to g while the scale change Δs_f is the s measure of the cost to f so the difference is a measure in terms of s of the net benefit of the project dY:

$$\Delta s = \Delta s_g - \Delta s_f > 0.$$

That is the increase in the size of the s pie due to the project dY. Now we use the MRS reasoning to compute the s-results of implementing both the project dY and the compensation dX. The scale change that counterbalances the cost to f is:

$$-\nabla f(X,Y)(dX,dY)/\nabla f(X,Y)(dX,0) = \Delta s_f - 1$$

while the scale change necessary to measure the gain to g is:

$$-\nabla g(X,Y)(dX,dY)/\nabla g(X,Y)(dX,0) = \Delta s_g - 1$$

so the net benefit of implementing both the project and the compensation is:

$$(\Delta s_g - 1) - (\Delta s_f - 1) = \Delta s_g - \Delta s_f + (1 - 1) = \Delta s.$$

Thus it is said in CB analysis that paying the compensation dX does not change the total benefit of the project dY (measured in terms of the scale parameter s for dX taken as the numeraire).

But dX does change the distribution of the benefits between f and g. When the project dY is undertaken and only the cost Δs_f is paid, then all the benefit accrues to g. If dY is undertaken and the maximum cost of Δs_g is paid then all the benefit accrues to f. But when the project dY and the compensation dX (s = 1) are both undertaken, then the increase in g is $\Delta s_g - 1$ while the increase in f (also measured in s) is $1 - \Delta s_f$, and the net benefit is the sum of the two parts:

$$\Delta s = (\Delta s_g - 1) + (1 - \Delta s_f).$$

In this inverted description, it is the change dY that accounts for the increase in the size of the pie measured in terms of the numeraire s while the change dX leaves the size of the s pie unchanged. Hence CB analysis would recommend the change dY on efficiency grounds while leaving dX as a separate equity decision. Yet this is exactly the same dX and dY changes where "efficiency" made the opposite recommendation based on the first description where t was taken as the numeraire. Since the efficiency recommendation reverses itself under a mere change in numeraire to describe exactly the same underlying situation, that recommendation cannot be sustained.

There are no efficiency grounds for recommending either dX or dY by itself. The mistake comes in trying to evaluate the changes using either of the changes itself as numeraire, i.e., the scale factor t for dY or the scale factor s for dX. Then the reasoning is vulnerable to "numeraire illusion" since it will always appear that the change in the numeraire itself has no effect on the size of the pie measured in terms of that same numeraire. That problem in the reasoning is revealed by reversing the numeraires.

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