On the Structure of Some Measures of Deprivation*

ACHIN CHAKRABORTY
Centre for Development Studies
Prasanthanagar Road, Ulloor, Thiruvananthapuram 695011
Kerala, India
E-mail: achin@cds.ac.in
PRASANTA K. PATTANAIK

Department of Economics, University of California
Riverside, CA 92521, U.S.A.
Telephone: 1 909 787 5037
Fax: 1 909 787 5685
Email: ppat@ucrac1.ucr.edu
YONGSHENG XU

Department of Economics, Andrew Young School of Policy Studies
Georgia State University, Atlanta, GA 30303, U.S.A.
Telephone: 1 404 651 2769
Fax: 1 404 651 4985
Email: yxu3@gsu.edu

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Abstract. This paper discusses the structure of some measures of deprivation. We

axiomatically characterise the mean of squared deprivation gaps, which is widely used in the applied literature and is a distinguished member of the important class of deprivation measures first introduced by Foster, Greer and Thorbecke (1984). In the course of doing so, we also provide, in an integrated framework, axiomatic characterisations of the Foster-Greer-Thorbecke class of deprivation measures and several conspicuous subclasses of this class.

JEL Classification Numbers: D3, I3, O1

Introduction

In recent years, there has been an increased interest in poverty and deprivation in general and their measurement in particular. The purpose of this paper is to explore the structure of some well-known measures of deprivation and to study them systematically in a unified framework. While these measures have been widely used in empirical research on deprivation, the structural features of some of them do not seem to have been investigated in detail. A conspicuous example of such a deprivation measure is what we shall call the *mean of squared deprivation gaps*. The mean of squared deprivation gaps, which measures social deprivation by the mean of the squares of the individuals' normalised deprivation levels, has been discussed extensively and used in many contexts in applied work (see, for example, Greer and Thorbecke (1986), Datt and Ravallion (1992, 1997), and Dutta and Pattanaik (2000)). The popularity of this measure is due to its many attractive features including its simplicity, its additive decomposability, and its sensitivity to poverty aversion (see Zheng (2000) for some discussions of the issue of sensitivity to poverty aversion). However, despite its importance and prominence in the literature on poverty and deprivation, there does not seem to be any comprehensive investigation of the properties of the mean of squared deprivation gaps. One of our main objectives is to investigate the structure of the mean of squared deprivation gaps by presenting an axiomatic characterisation for it.

The mean of squared deprivation gaps is a member of the class of deprivation measures that was first introduced by Foster, Greer, and Thorbecke (1984). Each deprivation measure in the Foster-Greer-Thorbecke (FGT) class, which also includes the 'head-count ratio' and the 'average deprivation gap', is the mean of certain powers of the individuals' normalised deprivation levels. Thanks to the property of additive decomposability, which every member of the FGT class of deprivation measures satisfies, the FGT class has occupied a unique and very important place in the analytical literature on the measurement of deprivation as well as in the applied work on poverty and deprivation. In the process of characterising the mean of squared deprivation gaps, we also provide axiomatisations of the FGT class of deprivation measures and various conspicuous subclasses of this class. Thus, we develop a unified framework to analyse and understand the mean of the squared deprivation gaps as well as the FGT class of deprivation measures and several important subclasses of this class.

In the literature, there are some attempts to characterise the FGT class of deprivation measures. Foster and Shorrocks (1991) provide an axiomatic characterisation for deprivation measures that are monotonic transformations of members of the FGT class of deprivation measures. Also, Ebert and Moyes (2002) present an axiomatisation of the class of poverty ordering that are representable by members of the FGT class of deprivation measures excluding the head-count ratio. Our contribution differs from these existing results in several aspects. First, as we mentioned earlier, the mean of squared deprivation gaps constitutes a major focus of our analysis. Neither Foster and Shorrocks nor Ebert and Moyes study this specific important measure of deprivation. Secondly, in the course of studying the features of the mean of squared deprivation gaps, we also develop a unified and integrated framework and use it to provide axiomatic characterisations of several important subclasses of the FGT class of deprivation measures as well as a characterisation of the FGT class itself. Thirdly, unlike Ebert and Moyes (2002), who start with a poverty ordering, we start with a deprivation index and impose restrictions on the deprivation index itself to derive our results. In contrast to the ordinal framework of Ebert and Moyes, we use a framework where the deprivation index is taken to be cardinal. While such cardinalism constitutes a stronger structural assumption, we believe that, in real life, we often articulate intuitions about poverty and deprivation, which have a cardinal basis. Thus, when one says that the reduction of poverty in a country during the 1990's was greater as compared to the 1980's, the intuition underlying the statement is essentially cardinal in nature. It is also our belief that some key properties of deprivation index would make more intuitive sense in a cardinal framework than in an ordinal framework. For example, the type of transfer sensitivity axiom introduced in Ebert and Moyes (2002), which is a variant of Kakwani's (1980) transfer sensitivity axiom, essentially requires comparisons of the difference in poverty between two suitably specified situations, say, A and B, and the difference in poverty between two other situations, say, C and D. For such comparisons, a cardinal framework for the measurement of deprivation would seem to be more appropriate than a purely ordinal framework.

The remainder of the paper is organised as follows. In Section 2, we introduce some basic notation and definitions. Section 3 introduces some axioms and presents an axiomatic characterisation of the FGT class of deprivation measures. In Section 4, we introduce some further axioms, and, using these new axioms together with the axioms introduced in Section 3, we provide axiomatic characterisations of the mean of squared deprivation gaps and of various important subclasses and members of the FGT class of deprivation measures including a joint characterisation of the head-count ratio, the average deprivation gap and the mean of squared deprivation gaps. We conclude in Section 5 by summarising our main results. The proofs of our results are given in Appendices A and B.

The Basic Notation and Definitions

Let $_+$ be the non-negative real line, and $^n_+$ be the n-dimensional, non-negative Euclidean space. For all $e \in {}^n_+$, all $i \in \{1,\ldots,n\}$, and all $a \in {}_+$, e_{-i} denotes the vector $(e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n)$, and (e_{-i},a) denotes the vector $(e_1,\ldots,e_{i-1},a,e_{i+1},\ldots,e_n)$. For all $c \in [0,1]$, let $c^{[n]}$ denote the n-dimensional vector with c for each of its components (in particular, we shall use the notation $0^{[n]}$ and $1^{[n]}$).

Consider a society consisting of n ($\infty > n \geq 4$) individuals. In what follows, we treat n as fixed. Let N denote $\{1,\cdots,n\}$. For all $i \in N$, let $z_i \in {}_{+}$ denote the 'achievement' of individual i, and let $\underline{z}_i \in {}_{+}$ denote the 'deprivation benchmark for i': i is deprived if and only if $\underline{z}_i > z_i$. It is often assumed that $\underline{z}_i$ is the same for all $i \in N$. However, in some contexts, one may like to allow $\underline{z}_i$ to take different values depending on the individual under consideration: for example, when fixing the income poverty line, one may like to have different values of $\underline{z}_i$ depending on whether the individual lives in an urban area or a rural area.

The notion of the achievement of an individual, as well as the notion of the deprivation benchmark, can have different interpretations. If one's concern is with the measurement of income poverty, an individual's achievement, as well as the benchmark for her, will be interpreted in terms of income. On the other hand, one can think of z_i as an index of i's overall achievement in terms of several 'real' indicators such as housing, health, education, etc.; in that case, $\underline{z}_i$ is to be interpreted as the minimum level of overall achievement of i that is considered satisfactory. For all $i \in N$, define x_i as follows

$$x_i = \begin{cases} 0 & z_i \geq \underline{z}_i \\ (\underline{z}_i - z_i)/\underline{z}_i & z_i < \underline{z}_i \end{cases}$$

Thus, we have an n-tuple vector $x = (x_1,\cdots,x_n)$, which we will call the *normalised deprivation vector* and which shows each individual's normalised deprivation level. Clearly, $x \in [0,1]^n$. For all $x,y \in [0,1]^n$, and all $N'(\phi \neq N' \subseteq N)$, we say that x and y are N'-variants iff [for all $i \in N', x_i \neq y_i$, and , for all $j \in N - N', x_j = y_j$].

A *deprivation measure* for the society is defined as a function $f : [0,1]^n \to {}_{+}$. For all $x,y \in [0,1]^n, f(x) \geq f(y)$ is interpreted as indicating that the degree of deprivation in situation x is at least as great as the degree of deprivation in situation y. Throughout this paper, we assume that the deprivation measure is cardinal in the sense of being unique up to an affine transformation.

Definition 2.1. Let f be a deprivation measure.

(i) f is a member of the *Foster-Greer-Thorbecke (FGT) class of deprivation measures* iff, for some non-negative real number α,

$$f(x) = \frac{1}{n} \sum_{i \in N} x_i^{\alpha}, x \in [0,1]^n.$$

(ii) f is the *mean of squared deprivation gaps* (MSDG) iff, for all $x \in [0,1]^n$,

$$f(x) = \frac{1}{n} \sum_{i \in N} x_i^2.$$

(iii) f is the *average deprivation gap* iff, for all $x \in [0,1]^n$,

$$f(x) = \frac{1}{n} \sum_{i \in N} x_i.$$

(iv) f is the *head-count ratio* iff, for all $x \in [0,1]^n$,

$$f(x) = \frac{\#\{i \in N \mid x_i > 0\}}{n}$$

The FGT class of deprivation measures, the mean of squared deprivation gaps, the average deprivation gap and the head-count ratio are well-known in the literature and hardly need explanation. Clearly, the FGT class of deprivation measures includes the mean of squared deprivation gaps ($\alpha = 2$), the average deprivation gap ($\alpha = 1$), and the head-count ratio

($\alpha = 0$).

The FGT Class of Deprivation Measures

In this section, we characterise the FGT class of deprivation measures.

Some properties of a deprivation measure

The following definition introduces the properties that we shall use to characterise the FGT class of deprivation measures.

Definition 3.1. Let f be a deprivation measure. We define the following properties of f.

(i) *Continuity* (CON). $f(x)$ is continuous in $(0,1]^n$.

(ii) *Normalisation* (NOR). $f(0^{[n]}) = 0, f(1^{[n]}) = 1$.

(iii) *Independence* (IND). For all $x, y \in [0,1]^n$, all $i \in N$, and all $t \geq 0$, if $x_i + t \in [0,1]$, then $f(x_{-i}, x_i + t) - f(x_{-i}, x_i) = f(y_{-i}, x_i + t) - f(y_{-i}, x_i)$.

(iv) *Uniform Scale-invariance* (USI). Let $x, y, x', y' \in [0,1]^n$. Suppose $f(x) - f(y) = f(x') - f(y')$. Then $f(kx) - f(ky) = f(kx') - f(ky')$ for every $k > 0$, such that $kx,\ ky,\ kx',\ ky' \in [0,1]^n$.

(v) *Anonymity* (AN). Let σ be a one-to-one function from N to N. Then, for all $x \in [0,1]^n, f(x) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

(vi) *Weak Monotonicity* (WMON). For all $x \in [0,1]^n$, all $i \in N$, and all $x'_i \in [0,1]$, if $x'_i > x_i$ then $f(x_{-i}, x'_i) \geq f(x)$. Further, $f(x) > f(0^{[n]})$ for all $x \in [0,1]^n$ such that $x \neq 0^{[n]}$.

CON is a technical requirement: the index function, f, is continuous in $(0,1]^n$. It requires that a 'small' change in the deprivation vector should cause only a 'small' change in the deprivation measure. It should be noted that the head-count ratio is not continuous at the zero deprivation vector. Therefore, in formulating continuity property, we use the half-open, half-closed interval $(0,1]$. NOR is a normalisation axiom and requires that: (1) the social deprivation index be 0 when every individual's normalised deprivation index is 0, that is, when the level of each individual's achievement is at or above the benchmark level of achievement; and (2) the social deprivation index be 1 when every individual's normalised deprivation index is 1. Given that, in our framework, the deprivation measure is interpreted to be cardinal, NOR is a very natural property. IND ensures that the deprivation measure is additive (see Lemma 3.2). It can be shown that IND is formally equivalent to the counterpart, in our context, of Foster, Greer, and Thorbecke's (1984) well-known property of "additive decomposability with population share weights". Its intuitive appeal is thus clear.

Uniform scale invariance requires that, if the difference between $f(x)$ and $f(y)$ is the same as the difference between $f(x')$ and $f(y')$, then the difference between $f(kx)$ and $f(ky)$ must also be the same as the difference between $f(kx')$ and $f(ky')$, where k is some positive constant such that $kx,\ ky,\ kx',\ ky' \in [0,1]^n$. The intuition of USI can be explained as follows. Consider two situations with respective normalized deprivation vectors x and y. Suppose there is an equiproportionate increase in the 'miseries' of all the people in both situations. Then the difference between the levels of 'social misery' in the two situations will change by an amount that will depend exclusively on the initial difference in 'social misery' and the proportionality factor by which the individual 'miseries' change. Anonymity is the very plausible restriction that the normalised deprivations of all individuals be treated symmetrically by f. Weak monotonicity stipulates that: (1) if the normalised deprivation of an individual increases, the normalised deprivations of other individuals remaining the same, then the deprivation of the society cannot decrease; and (2) when the achievement of any individual falls short of her benchmark level, the society's deprivation is higher than what it would be if no individual's achievement fell short of her benchmark level.

An axiomatic characterisation of the FGT class

We proceed via the following two lemmas to our axiomatic characterisation of the FGT class of deprivation measures.

Lemma 3.2. A deprivation measure f satisfies IND if and only if

(3.1) there exist functions $f_i : [0,1] \to {}_+$ $(i = 1,2,\ldots,n)$, such that, for all $x \in [0,1]^n$, $f(x) = \sum_{i \in N} f_i(x_i)$.

Proof: The proof is given in Appendix A. ■

Lemma 3.3. Let $g : [0,1] \to {}_+$ be a function such that (i) it is continuous in $(0,1]$, (ii) $g(a) \geq g(b)$ for all $a,b \in [0,1]$ with $a \geq b$, and (iii) $g(a) > g(0) = 0$ for all $a \in (0,1]$. Suppose g satisfies the following condition:

(3.2) for all $a,b,a',b' \in (0,1]$, $[g(a) - g(b) = g(a') - g(b')]$ implies $g(ka) - g(kb) = g(ka') - g(kb')$ for all $k > 0$ such that $ka, kb, ka', kb' \in (0,1]$.

Then, for some constants $\theta, \alpha,$ and $\gamma,$ and for all $c \in (0,1]$, $g(c) = \theta c^{\alpha} + \gamma$.

Proof: The proof is given in Appendix A. ■

Now, we present the characterisation of the class of FGT deprivation measures.

Theorem 3.4. A deprivation measure f satisfies CON, NOR, IND, USI, AN and WMON if and only if f belongs to the class of FGT deprivation measures (see Definition 2.1).

Proof: The proof is given in Appendix A. ■

Some Subclasses and Members of the FGT Class

In this section, we analyse the structure of several important subclasses and members of the FGT class of deprivation measures

Some further properties of a deprivation measure

Definition 4.1. Let f be a deprivation measure.

(i) *Monotonicity* (MON). For all $x \in [0,1]^n$, all $i \in N$, and all $x_i' \in [0,1]$, if $x_i' > x_i$ then $f(x_{-i}, x_i') > f(x)$.

(ii) *Transfer Axiom* (TA). For all $x,y \in [0,1]^n$, all distinct $i,j \in N$, and all $\delta \in (0,1]$, if [x and y are $\{i,j\}$ −variants], $[x_i > x_j]$, $[y_i = x_i + \delta]$, and $[y_j = x_j - \delta]$, then $f(y) > f(x)$.

(iii) *Transfer Sensitivity* (TS). For all $x,x',y,y' \in [0,1]^n$, all distinct $i,j \in N$, and all $\delta, t > 0$, if [for all $a,b \in \{x,y,x',y'\}$, $a_k = b_k$ for all $k \in N - \{i,j\}$], $[x_i = x_j + \delta$ and $x_j' = x_j - t$ and $x_i' = x_i + t]$, and $[y_i = y_j + \delta$ and $y_j' = y_j - t$ and $y_i' = x_i + t]$, then $[y_i > x_i]$ implies $[f(y') - f(y) > f(x') - f(x)]$.

(iv) *Equivalent Transfer* (ET). Let $x,y,w \in (0,1]^n$, $i,j,i',j' \in N$, and $t \in [0,1]$ be such that $[x_i - x_j = x_{i'} - x_{j'}]$, [$x$ and y are $\{i,j\}$ −variants and x and w are $\{i',j'\}$ −variants], $[y_i = x_i - t$ and $y_j = x_j + t]$, and $[w_{i'} = x_{i'} - t$ and $w_{j'} = x_{j'} + t]$. Then $f(y) = f(w)$.

MON, which is a stronger property than WMON, has its conceptual origin in Sen(1976). It requires that the society's deprivation must increase if, other things remaining the same, some individual's normalised deprivation increases. The transfer axiom requires that if, initially, individuals *i* and *j* are both deprived, but the normalised deprivation of *i* is higher than that of *j*, then, other things remaining the same, an increase of δ in the normalised deprivation of *i*, together with a decrease of δ in the normalised deprivation of *j*, will increase the deprivation of the society. Our transfer axiom is reminiscent of (though, formally, not identical to) Sen's (1976) transfer axiom which stipulates that a transfer of income from a poor person to anybody richer must increase the society's poverty. TS requires that, if, initially, individuals *i* and *j* are both deprived, but the normalised deprivation of *i* is higher than that of *j* by an amount $\delta > 0$, and, if, subsequently, other things remaining the same, the normalised deprivation of the less deprived individual *j* decreases by t ($t > 0$) and, simultaneously, the normalised deprivation of the more deprived individual *i* increases by t, then the resultant increase in the social deprivation must be greater the larger the magnitude of *i*'s initial normalised deprivation. Our transfer sensitivity axiom is similar to Kakwani's (1980) transfer sensitivity axiom which stipulates that, when a positive amount of income from a poor individual with income z_i is transferred to a poor individual with income $z_i + a > z_i$, the consequent increase in the poverty of the society must be larger the smaller z_i is. ET can be explained as follows. Consider four individuals i, j, i', and j', and a deprivation vector x, such that everybody is deprived in x and the normalised deprivation of *i* exceeds (resp. falls short of) the normalised deprivation of *j* by exactly the same amount by which the normalised deprivation of i' exceeds (resp. falls short of) the normalised deprivation of j'. Now, consider two alternative changes, starting with the initial situation of x. In the first case, other things remaining the same, the normalised deprivation of *i* decreases by t ($t \geq 0$), the normalised deprivation of *j* increases by t, and *i* continues to be deprived after the change. In the second case, the normalised deprivation of i' decreases by t, the normalised deprivation of j' increases by t, and i' continues to be deprived after the change. Then ET requires that the new level of social deprivation in the first case must be the same as the new level of social deprivation in the second case. Thus, given that in the initial situation the 'deprivation difference' between *i* and *j* is the same as the 'deprivation difference' between i' and j', ET ensures that the change in social deprivation when normalised deprivation of t is 'transferred' from *i* to *j* is the same as the change in social deprivation when normalised deprivation of t is 'transferred' from i' to j'.

The following lemma notes some relations among the properties introduced in Definitions 3.1 and 4.1.

Lemma 4.2.

(i) MON implies WMON.
(ii) If a deprivation measure satisfies ET, then it must violate TS.
Proof: See Appendix B for the proof. ■

$\alpha > 0; \alpha > 1;$ **and** $\alpha > 2$

It is clear that the head-count ratio ($\alpha = 0$) is the only member of the FGT class that violates MON. Therefore, the following result follows immediately from Theorem 3.4.

Theorem 4.3. A deprivation measure satisfies the properties CON, NOR, IND, USI, AN, and MON if and only if

(4.1) for some $\alpha > 0$, $f(x) = \frac{1}{n}\sum_{i\in N} x_i^{\alpha}$, for all $x \in [0,1]^n$.

When we add the further property of TA to CON, NOR, IND, USI, AN and MON, we get a characterisation of the subclass of the FGT class that corresponds to $\alpha > 1$.

Theorem 4.4. A deprivation measure satsfies CON, NOR, IND, USI, AN, MON and TA if and only if

(4.2) for some $\alpha > 1$, $f(x) = \frac{1}{n}\sum_{i\in N} x_i^{\alpha}$, for all $x \in [0,1]^n$.

Proof: See Appendix B for the proof. ■

Referring to the mean of squared deprivation gaps, Greer and Thorbecke (1986) remarked that it "is the only measure of which we are aware which satisfies the monotonicity and transfer axioms and is decomposable". As our Theorem 4.4 shows, all deprivation measures satisfying (4.2) (the mean of squared deprivation gaps is just one such measure) satisfy monotonicity, the transfer axiom, and "decomposability".

Next we consider what happens when we stipulate that a deprivation measure must satisfy TS, in addition to CON, NOR, IND, USI, AN, MON and TA. The following theorem summarises our finding.

Theorem 4.5. A deprivation measure satsfies CON, NOR, IND, USI, AN, MON, TA and TS if and only if

(4.3) for some $\alpha > 2$, $f(x) = \frac{1}{n}\sum_{i\in N} x_i^{\alpha}$, for all $x \in [0,1]^n$.

Proof: See Appendix B for the proof. ■

The head-count ratio, the average deprivation gap, and the mean of squared deprivation gaps

Three members of the FGT class, namely, the head-count ratio ($\alpha = 0$), the average deprivation gap ($\alpha = 1$), and the mean of squared deprivation gaps ($\alpha = 2$), have received much attention in the applied literature. We now consider these three measures.

Theorem 4.6. A deprivation measure satisfies CON, NOR, IND, USI, AN, WMON and ET if and only if it is the head-count ratio or the average deprivation gap or the mean of squared deprivation gap.

Proof: See Appendix B for the proof. ■

Since MON implies WMON (see Lemma 4.2), the head-count ratio violates MON, and both the average deprivation gap and the mean of squared deprivation gaps satisfy MON (see Theorem 4.3), Theorem 4.7 below follows immediately from Theorem 4.6.

Theorem 4.7. A deprivation measure satisfies CON, NOR, IND, USI, AN, MON and ET if and only if it is either the average deprivation gap or the mean of squared deprivation gaps.

Our final result provides an axiomatic characterisation of the mean of squared deprivation gaps, which is widely used in the empirical literature.

Theorem 4.8. A deprivation measure satisfies CON, NOR, IND, USI, AN, MON, ET and

TA if and only if it is the mean of squared deprivation gaps.

Proof: The proof is given in Appendix B. ■

Concluding Remarks

In this paper, we have presented a unified framework to study the structure of some well-known and important measures of normalised deprivation by providing axiomatic characterisations for them. Our main results are summarised in the following table.

Insert Table 1

We hope that our analysis will lead to a better understanding of the structure of these measures and give these measures a firmer theoretical foundation.

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Table 1

The class of deprivation measures	**Properties characterising the class of deprivation measures**
The FGT class: $f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha \geq 0$	CON, NOR, IND, USI, AN, and WMON
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha > 0$	CON, NOR, IND, USI, AN, and MON
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha > 1$	CON, NOR, IND, USI, AN, MON, and TA
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha > 2$	CON, NOR, IND, USI, AN, MON, TA, and TS
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha \in \{0,1,2\}$	CON, NOR, IND, USI, AN, WMON, and ET
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{\alpha}$, $\alpha \in \{1,2\}$	CON, NOR, IND, USI, AN, MON, and ET
$f(x) = \frac{1}{n}\Sigma_{i\in N}x_i^{2}$	CON, NOR, IND, USI, AN, MON, ET, and TA

Appendix A

Proof of Lemma 3.2. Suppose f satisfies IND. Let $x \in [0,1]^n$.

$$\begin{aligned} f(x) = {} & [f(x_1,x_2,\ldots,x_n) - f(0,x_2,\ldots,x_n)] \\ & + [f(0,x_2,\ldots,x_n) - f(0,0,x_3,\ldots,x_n)] \\ & + \ldots \\ & + [f(0,0,\ldots,0,x_n) - f(0^{[n]})] \\ & + f(\,0^{[n]}). \end{aligned}$$

By IND, it follows that

$$\begin{aligned} f(x) = {} & [f(0^{[n]}_{-1},x_1) - f(\,0^{[n]})] \\ & + [f(0^{[n]}_{-2},x_2) - f(\,0^{[n]})] \\ & + \ldots \\ & + [f(0^{[n]}_{-n},x_n) - f(\,0^{[n]})] \\ & + f(\,0^{[n]}) \\ = {} & f_1(x_1) + \ldots + f_n(x_n), \end{aligned}$$

where, for every $i \in N$, $f_i(x_i) = [f(0^{[n]}_{-i},x_i) - \frac{n-1}{n} f(\,0^{[n]})]$.

It is obvious that, if f satisfies (3.1), then f must satisfy IND.

Proof of Lemma 3.3. Let $g : [0,1] \to {}_+$ be a function such that it is continuous in $(0,1]$, $g(a) \geq g(b)$ for all $a,b \in [0,1]$, and $g(a) > g(0)$ for all a$\in (0,1]$ and let g satisfy (3.2). If g is constant in $(0,1]$, then it is clear that there exist constants θ and γ such that, for all $c \in (0,1], g(c) = \theta c^0 + \gamma$. In what follows, we discuss the situation in which g is not constant in $(0,1]$.

For the sake of clarity, we divide into several successive steps the proof for the case where g is not constant in $(0,1]$.

Step 1. For all $c \in {}_+$ with $c \neq 0$, c can be written as $c = p/q$ where $p,q \in (0,1]$. Define

$$G(c) = \frac{g(p) - g(0)}{g(q) - g(0)}(g(1) - g(0)) + g(0), c > 0, c = \frac{p}{q}, p,q \in (0,1]$$

and

$$G(0) = g(0).$$

Note that, since $g(a) > g(0)$ for all $a \in (0,1], G(c) \geq g(0) = G(0)$ for all $c \in {}_+$.

Step 2. We show that the function G, introduced in the preceding step, is well defined on ${}_+$. Let $c_1, c_2 \in {}_+$ with $c_1 = c_2$. If $c_1 = c_2 = 0$, then $G(c_1) = G(c_2) = 0$. Otherwise, let $c_1 = p_1/q_1$ and $c_2 = p_2/q_2 = (q_1p_2/q_2)/q_1$ for some $p_1,q_1,p_2,q_2 \in (0,1]$. Since $c_1 = c_2$, clearly, $q_1p_2/q_2 = q_1p_1/q_1 = p_1 \leq 1$. Then,

$$G(c_1) = \frac{g(p_1) - g(0)}{g(q_1) - g(0)}(g(1) - g(0)) + g(0),$$

and

$$G(c_2) = \frac{g(q_1p_2/q_2) - g(0)}{g(q_1) - g(0)}(g(1) - g(0)) + g(0).$$

Note that $p_1 = q_1p_2/q_2$. g being a function, it follows that $g(p_1) = g(q_1p_2/q_2)$. Therefore, $G(c_1) = G(c_2)$. Thus, G is well defined on ${}_+$.

Step 3. Now we show that G satisfies the following:

(A.1) for all positive real numbers $u, v, \underline{u}, \underline{v}$, $[G(u) - G(v) = G(\underline{u}) - G(\underline{v})]$ implies $G(ku) - G(kv) = G(k\underline{u}) - G(k\underline{v})$ for all $k > 0$.

Let $u, v, \underline{u}, \underline{v}$ be positive real numbers such that $G(u) - G(v) = G(\underline{u}) - G(\underline{v})$. Let $u = p_1/q_1, v = p_2/q_2, \underline{u} = p_3/q_3, \underline{v} = p_4/q_4$, where, for all $t \in \{1,2,3,4\}$, $p_t \in (0,1]$ and $q_t \in (0,1]$. Without loss of generality, let $u \geq v, u \geq \underline{u}, u \geq \underline{v}$. Then, $p_1 \geq q_1 p_t/q_t$ and $p_1/k \geq q_1 p_t/kq_t$ where $t = 2,3,4$ and $k \geq 1$. From

$$G(u) - G(v) = G(\underline{u}) - G(\underline{v}),$$

we obtain

$$g(p_1) - g(q_1 p_2/q_2) = g(q_1 p_3/q_3) - g(q_1 p_4/q_4),$$

and

$$g(p_1/k) - g(q_1 p_2/kq_2) = g(q_1 p_3/kq_3) - g(q_1 p_4/kq_4),$$

where $k \geq 1$. Since g satisfies (3.2), it must be true that

$$g(kp_1) - g(kq_1 p_2/q_2) = g(kq_1 p_3/q_3) - g(kq_1 p_4/q_4),$$

for all $k > 0$ and $k < 1$, and

$$g(kp_1/k) - g(kq_1 p_2/kq_2) = g(kq_1 p_3/kq_3) - g(kq_1 p_4/kq_4),$$

for all $k \geq 1$. By the definition of G, we then obtain

$$G(ku) - G(kv) = G(k\underline{u}) - G(k\underline{v}), k > 0.$$

Thus, G satisfies $(A.1)$.

Step 4. It can be checked that $G(c)$ is continuous for all $c > 0$. G is nonconstant since g is nonconstant in $(0,1]$. Since G satisfies (A.1) and is continuous for positive numbers and nonconstant, by the result of Eichhorn and Gleissner (1988) and Aczél (1988), (A.2) below must be true of G :

(A.2) either [for some constants θ, α and γ, $G(c) = \theta c^{\alpha} + \gamma$, for all $c > 0$) or (for some constants θ and γ, $G(c) = \theta \log c + \gamma$, for all $c > 0$).

Step 5. G is an extension of g, since, $G(0) = g(0)$, and for all $c \in (0,1]$, we have

$$G(c) = G(c/1) = \frac{g(c) - g(0)}{g(1) - g(0)}(g(1) - g(0)) + g(0) = g(c).$$

From the definition of G, the extension G of g is unique. G being an extension of g, noting (A.2), it follows that

(A.3) either (for some constants θ, α and γ, $g(c) = \theta c^{\alpha} + \gamma$, for all $c \in (0,1]$) or (for some constants θ and γ, $g(c) = \theta \log c + \gamma$, for all $c \in (0,1]$).

Since $g(a) > g(0) = 0$ for all $a \in (0,1]$, and $g(a) \geq g(b)$ for all $a, b \in (0,1]$ with $a \geq b$, g cannot be such that $g(c) = \theta \log c + \gamma$. Therefore, when g is nonconstant in $(0,1]$, it must be true that, for some constants θ, α, γ, $g(c) = \theta c^{\alpha} + \gamma$ for all $c \in (0,1]$. This completes the proof for the case where g is not constant in (0, 1] (recall that the case where g is constant in (0, 1] has been covered earlier).

Proof of Theorem 3.4. It is straightforward to show the necessity of the properties for *f* to belong to the FGT class of deprivation measures. We give the proof for the sufficiency of these properties for f to belong to the FGT class. Let f satisfy CON, NOR, IND, USI, AN and WMON.

Given IND, by Lemma 3.2,

(A.4) for all $x \in [0,1]^n, f(x) = \sum_{i \in N} f_i(x_i)$,

where for every $i \in N$, f_i is a function from [0, 1] to $_+$. It is clear that, by AN, the function f_i must be the same for all $i \in N$. Let g denote the function f_i for all $i \in N$. Then

(A.5) for all $x \in [0,1]^n, f(x) = \sum_{i \in N} g(x_i)$.

By NOR, from (A.5), $g(0) = 0$ follows easily.

Let $a, b, a', b' \in (0,1]$ and $k > 0$ be such that $g(a) - g(b) = g(a') - g(b')$ and $ka, kb, ka', kb' \in (0,1]$. Then, by (A.5), we must have $f(a^{[n]}) - f(b^{[n]}) = f(a'^{[n]}) - f(b'^{[n]})$, and hence, by USI, $f(ka^{[n]}) - f(kb^{[n]}) = f(ka'^{[n]}) - f(kb'^{[n]})$. By (A.5), it follows that $g(ka) - g(kb) = g(ka') - g(kb')$. Thus, we have shown that g satisfies (3.2).

By WMON, for all $c \in (0,1]$, $f(c^{[n]}) > f(0^{[n]})$. Therefore, by (A.5), we have

(A.6) $g(c) > g(0)$ for all $c \in (0,1]$.

Since f is continuous in $(0,1]^n$, g is continuous in $(0,1]$. By WMON, from (3.2), it follows that $g(a) \geq g(b)$ for all $a, b \in (0,1]$ with $a \geq b$. Now, since g is continuous in $(0,1]$ and satisfies (3.2) and (A.6), by Lemma 3.3,

(A.7) for all $i \in N$, there exist constants θ, α, and γ such that, for all $a \in (0,1], g(a) = \theta a^\alpha + \gamma$.

Noting that $g(1) \geq g(c) > g(0)$ for all $c \in (0,1]$, and θ and γ are given constants, it can be checked that $\alpha \geq 0$. Then, noting that $g(0) = 0$, for all $x \in [0,1]^n$,

$$f(x) = \sum_{i \in N} [\theta x_i^\alpha + \gamma] = \theta \sum_{i \in N} x_i^\alpha + n\gamma\alpha > 0,$$

and

$$f(x) = \theta \#\{i \in N | x_i > 0\} + n\gamma\alpha = 0.$$

For $\alpha > 0, f(x) = \theta \sum_{i \in N} x_i^\alpha + n\gamma$ is continuous in $[0,1]^n$. Letting $x = 0^{[n]}$, by NOR, it follows that $\gamma = 0$. Then, letting $x = 1^{[n]}$, by NOR again, it follows that $\theta = \frac{1}{n}$. Thus, for $\alpha > 0$, for all $x \in [0,1]^n$, $f(x) = \frac{1}{n} \sum_{i \in N} x_i^\alpha$. For $\alpha = 0$, by IND, we have $f(1,0,\cdots,0) - f(0^{[n]}) = f(1,1,0,\cdots,0) - f(0,1,0,\cdots,0)$; that is, $\theta + n\gamma - 0 = 2\theta + n\gamma - (\theta + n\gamma)$. Noting that $n \geq 4$, we obtain $\gamma = 0$. Letting $x = 1^{[n]}$, by NOR, it follows that $\theta = \frac{1}{n}$. Thus, for $\alpha = 0$, for all $x \in [0,1]^n, f(x) = \frac{\#\{i \in N | x_i > 0\}}{n}$. Therefore, f belongs to the FGT class of deprivation measures.

Appendix B

Proof of Lemma 4.2.

(i) The proof is obvious from the definitions.

(ii) The proof follows from the definitions of ET and TS. ■

Proof of Theorem 4.4. By Theorem 4.3, it is enough to show that a deprivation measure satisfying (4.1) satisfies TA iff it satisfies (4.2). Let f be a deprivation measure satisfying (4.1). However, noting that f is twice differentiable in $(0,1]^n$, it is clear that f satisfies TA iff the second derivative of f with respect to each of its arguments is positive everywhere in $(0, 1]^n$. However, given that f satisfies (4.1), the second derivative of f with respect to each of its arguments is positive everywhere in $(0, 1]^n$ iff f satisfies (4.2). ■

Proof of Theorem 4.5. In view of Theorem 4.4, it is enough to prove that a deprivation measure satisfying (4.2) satisfies TS iff it satisfies (4.3). Let f satisfy (4.2), so that, for some $\alpha > 1$, $f(w) = \frac{1}{n}\sum_{i\in N} w_i^\alpha$, for all $w \in [0,1]^n$.

(i) Suppose f satisfies TS. Consider $x,x',y,y' \in [0,1]^n$, distinct $i,j \in N$, and $\delta, t > 0$ such that [for all $a,b \in \{x,y,x',y'\}$, $a_k = b_k$ for all $k \in N - \{i,j\}$], [$x_i = x_j + \delta$ and $x'_j = x_j - t$ and $x'_i = x_i + t$], and [$y_i = y_j + \delta$ and $y'_j = y_j - t$ and $y'_i = x_i + t$] and $y_i > x_i$. Then, by TS, $f(y') - f(y) > f(x') - f(x)$. It follows that [$y_i'^\alpha - y_i^\alpha + y_j'^\alpha - y_j^\alpha > x_i'^\alpha - x_i^\alpha + x_j'^\alpha - x_j^\alpha$], i.e.,

(B.2) $(y_i - t)^\alpha - y_i^\alpha + (y_i + \delta + t)^\alpha - (y_i + \delta)^\alpha > (x_i - t)^\alpha - x_i^\alpha + (x_i + \delta + t)^\alpha - (x_i + \delta)^\alpha$.

Then

(B.3) $\frac{1}{t}[(y_i - t)^\alpha - y_i^\alpha + (y_i + \delta + t)^\alpha - (y_i + \delta)^\alpha] > \frac{1}{t}$ $[(x_i - t)^\alpha - x_i^\alpha + (x_i + \delta + t)^\alpha - (x_i + \delta)^\alpha]$.

Note that (B.3) holds for all $t (t > 0)$. Then, letting t approach 0, and noting that $\alpha > 1$ and that the first derivative of w_i^α exists everywhere in [0, 1], we have

(B.4) $\alpha(y_i + \delta)^{\alpha-1} - \alpha y_i^{\alpha-1} > \alpha(x_i + \delta)^{\alpha-1} - \alpha x_i^{\alpha-1}$.

Noting $\alpha > 1$, it follows that

(B.5) $(y_i + \delta)^{\alpha-1} - y_i^{\alpha-1} > (x_i + \delta)^{\alpha-1} - x_i^{\alpha-1}$.

Since (B.5) holds for all positive δ, dividing both sides of (B.5) by δ and letting δ approach 0, we have

(B.6) $(\alpha - 1)y_i^{\alpha-2} > (\alpha - 1)x_i^{\alpha-2}$.

Hence, noting $\alpha > 1$,

(B.7) $y_i^{\alpha-2} > x_i^{\alpha-2}$.

Since (B.7) holds for all $y_i, x_i \in (0,1]$, such that $y_i > x_i$, it is clear that $\alpha > 2$, and hence f satisfies (4.2).

(ii) Now suppose f satisfies (4.2). Then the third derivative of f with respect to each of its arguments is positive everywhere in $(0,1]^n$. Then it can be easily shown that f must satisfy TS. ■

Proof of Theorem 4.6. In view of Theorem 3.4, it is enough to show that the only three members of the FGT class of deprivation measures that satisfy ET are the head-count ratio, the average deprivation gap and the mean of squared deprivation gaps.

Let f be a deprivation measure in the FGT class, so that, for some $\alpha \geq 0$, $f(x) = \frac{1}{n}\sum_{i\in N} x_i^\alpha$, for all $x \in [0,1]^n$. It can be easily checked that if f is the head-count ratio or the average deprivation gap or the mean of squared deprivation gaps, f must satisfy ET. We only need to show that, if f satisfies ET, then it must be one of these three deprivation measures. Suppose f satisfies ET. Then, with reasoning of the type that we used in proving Theorem 4.5, it can be shown that, for all $i \in N$, the third derivative of x_i^α is 0 everywhere in (0, 1]. Given that $\alpha \geq 0$, it is then clear that α can take only one of three values, 0, 1 and 2. This completes the proof. ■

Proof of Theorem 4.8. In view of Theorem 4.7, we only need to show that the average

deprivation gap violates TA while the mean of squared deprivation gaps satisfies it. That the average deprivation gap violates TA is obvious. To prove that the mean of squared deprivation gaps satisfies TA, consider $x, y, w \in (0,1]^n$, $i, j, i', j' \in N$, and $t \in [0,1]$ such that $[x_i - x_j = x_{i'} - x_{j'}]$, [$x$ and y are $\{i,j\}$ –variants and x and w are $\{i',j'\}$ –variants], [$y_i = x_i - t$ and $y_j = x_j + t$], and [$w_{i'} = x_{i'} - t$ and $w_{j'} = x_{j'} + t$]. Then,

(B.8) [

$y_i^2 + y_j^2 + y_{i'}^2 + y_{j'}^2 = (x_i - t)^2 + (x_j + t)^2 + x_{i'}^2 + x_{j'}^2 = x_i^2 + x_j^2 + x_{i'}^2 + x_{j'}^2 + 2t^2 - 2t(x_i - x_j)$ and $[w_i^2 + w_j^2 + w_{i'}^2 + w_{j'}^2 = x_i^2 + x_j^2 + (x_{i'} - t)^2 + (x_{j'} + t)^2 = x_i^2 + x_j^2 + x_{i'}^2 + x_{j'}^2 + 2t^2 - 2t(x_{i'} - x_{j'})]$.

Since, by assumption, $[x_i - x_j = x_{i'} - x_{j'}]$, and, for all $k \in N - \{i,j,i',j'\}, y_k = w_k$, from (B.8) it follows that $\sum_{k \in N} y_k^2 = \sum_{k \in N} w_k^2$. Hence the mean of squared deprivation gaps satisfies TA. ■