Group Efforts when Performance Is Determined by the "Best Shot"

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Abstract

The achievements of a group frequently depend on the efforts of just a few members but represent a public good benefit to all members. This paper investigates such a situation: the private provision of a public good whose level is determined as the *maximum* effort made by a group member. Members' costs of effort are envisioned as either commonly known or privately known. With perfect information, symmetric equilibria are in mixed strategies. For symmetric games, any number of players may be active and we characterize the unique equilibrium in which active contributors use the same strategy. An increase in the number of active players causes each active player to stochastically reduce his contributions and the distribution of the realized level of the public good shifts leftward, reducing equilibrium payoffs to inactive players. When information is private, we focus on the symmetric equilibrium, which is in pure strategies. Now an increase in the number of players yields a pointwise reduction in the equilibrium contribution strategy but an increase in equilibrium payoffs. Stochastically increasing the distribution of players' costs increases players' contribution strategy, with a resulting decrease in interim payoffs; and increasing *ex ante* heterogeneity of players' costs can increase payoffs. Whether information is public or private, equilibria are inefficient—we provide mechanisms that improve efficiency.

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1 Introduction

We study a situation in which the performance of a group is determined by the greatest effort of one of its members. Hirshleifer (1983) coined such a relationship a "best shot" production function.[1] This contrasts with the classic model of private public good provision by Bergstrom, Blume, and Varian (1986), in which the level of the public good depends on the *sum* of individuals' contributions.[2] As an example of a "best-shot" situation Hirshleifer suggested multiple modes of nuclear missile attack on an enemy target (only one missile needs to get through). The importance of the best-shot scenario is underscored by the many other examples that have been offered in the literature since Hirshleifer's seminal contribution, including:

- *Research and development.* For instance, see Arce and Sandler (2001), Chowdhury *et al.* (2011), Croson *et al.* (2006a), and in particular Sandler (1998), p. 231: "To find a cure for Ebola, AIDS, or antibiotic-resistant tuberculosis, the research with the greatest effort is typically the one that meets with success. For best-shot, whoever is first over the line wins for everyone. Thus, once a breakthrough is found for safely storing highly radioactive materials, the nation achieving this discovery determines the public good level of containment for everyone."

- *Military alliances.* For instance, see Conybeare *et al.* (1994), Sandler (1998), and in particular Sandler and Hartley (2001), p. 880: "A best-shot technology is relevant for a 'Star Wars' defense in which one or more allies possess sufficient defensive weapons to destroy an attacking nation's nuclear missiles shortly after launch. Nuclear deterrence is another instance of best shot. The U.S. nuclear arsenal was sufficient owing to its second-strike capability to deter a Soviet attack. British and French nuclear forces really did nothing to bolster this deterrence during the Cold War."

- *System reliability theory.* For instance, see Hausken (2002), Varian (2004), Danezis and Anderson (2005), and in particular Grossklags *et al.* (2008), pp. 211–212: "Censorship-resistant networks are another example of best shot games. A piece of information will remain available to the public domain as long as a single node serving that piece of information can remain unharmed."

Calvó-Armengol and Jackson (2010), p. 79, provide another category of situations that can be described as best-shot public good games: "...gathering information that is then freely communicated to other agents...." This category is especially interesting if one explicitly embeds it in a two-stage process:

1. A first stage of acquisition of the information or, moving the reasoning a further step back, a stage in which competences are developed or structures are built to facilitate information acquisition; and

[1] See also Hirshleifer (1985) and Harrison and Hirshleifer (1989) for expanded analysis and experimental evaluations.
[2] An early analysis of the relation among different public good production functions appears in Cornes (1993).

2. A second stage of information communication.

The example of the provision of a tsunami warning system of Kroll *et al.* (2007) fits well the above scheme: the information communication stage works with a best-shot technology (ideally), since the second report of an impending tsunami should find residents already scrambling for higher ground and thus adds no further utility to the first report. This best-shot character is inherited by the earlier, competence-development or structure-building phase: only the better-built and most sensitive tsunami spotting system will end up generating effective warnings. The workings of this example apply well beyond information gathering and communication. Indeed, to have a best-shot game in the competence-development or structure-building phase, it is not even necessary to have a best-shot technology in the following stage; what matters is that in the behavioral outcome of the following stage only the agent with the better technology provides a public good for the entire relevant population.

Finally, the variation of the best-shot public good game in which effort is dichotomous is the so-called "volunteer's dilemma." Examples include jumping on a live grenade, murdering a tyrant, cleaning toilets, chairing a department, volunteering entrepreneurial services, driving for a car pool, and slaying dragons.[3] This situation—where a volunteer's effort cost is fixed and the decision is simply to volunteer or not—has been studied extensively. Palfrey and Rosenthal (1984) studied this more generally where, say, m volunteers of n potential players are needed for the project to be a success. Harrington (2001) and Xu (2001) also studied these situations of volunteerism with a particular focus on how group size affects the likelihood that a volunteer comes forth (so that the public good is provided). In contrast to the volunteer's dilemma, we allow agents' efforts to be variable. Therefore, our setup is better suited to situations akin to those previously presented for instance under the research and development category, in which it is feasible to build research facilities of different sizes or qualities.

Our analysis is distinguished from the vast majority of the previously mentioned papers employing the best-shot technology (with variable, continuous inputs) in that we move beyond the asymmetric equilibria in which one player produces the public good for the entire group and everybody else abstains from exerting effort.[4] We do this for three reasons:

1. Best-shot public goods combine two kinds of market failure: the presence of public goods along with a non-convex production technology. Correspondingly, one may expect two sources of inefficiency to arise: free-riding and wasteful duplication of efforts. The second source of inefficiency disappears with a focus on equilibria in which all but on player provides no effort. Therefore, our going beyond these

[3] See, e.g., Bilodeau and Slivinsky (1996a, 1996b), Bilodeau *et al.* (2004), Bliss and Nalebuff (1984), Croson *et al.* (2006b), Diekmann (1985), Kirchgässner (2010), and Weesie (1994). The models of Bliss and Nalebuff (1984) and Bilodeau and Slivinsky (1996a) stand out because they add a timing dimension to the volunteer's dilemma, using a war-of-attrition framework.

[4] Arce and Sandler (2001) also go beyond this equilibrium by focusing on correlated equilibria.

equilibria gives an alternative, albeit more negative, accounting of the inefficiencies that best-shot public goods may generate.

2. The asymmetric equilibrium with only one player exerting effort is typically independent of interesting parameters of the model. Moreover, a strategy profile in which only one player provides the public good may not be an equilibrium if there is private information.

3. The equilibrium in which one player produces the public good for the entire group and everybody else abstains apparently requires some degree of coordination among players, especially if the game is symmetric. Coordination may be problematic if the best-shot public good game arises at the competence-development or structure-building stage, because at that stage agents may not even know who the other members of their group will be. For instance, consider a married couple in which only one member is in charge of preparing the joint tax return, because he or she is "better at it." The competence-development phase of tax-preparation abilities takes place, at least in part, over several years, at a time in which the members of the now married couple are not likely to have met yet.[5]

We assume players have a common increasing, concave benefit function for the public good and we investigate two scenarios, focusing primarily on symmetric environments. First, we consider players with a common, known marginal cost of effort. Any equilibrium having at least two active players (those with a positive probability of contributing toward the good) has those players use mixed strategies.[6] As the number of active players increases, players stochastically reduce their efforts, and they do so sufficiently to shift downward the distribution of the "best shot." These equilibria can be Pareto ranked, with better equilibria having fewer active players. We also show how the equilibrium distribution of effort depends on the risk aversion exhibited by the benefit function for the public good.

Second, we consider private information about players' marginal costs. We characterize the symmetric Bayesian equilibrium and provide comparative statics results with respect to the distribution of a player's marginal cost of effort. Increasing the number of members of the group causes equilibrium strategies to decrease (pointwise), while increasing players' interim payoffs. An increase in the distribution of costs (in the sense of first-order stochastic dominance) causes players to increase (pointwise) their strategies, with a concomitant reduction in interim payoffs. We also provide conditions under which reducing the riskiness of the distribution of costs reduces both *ex ante* and interim expected payoffs. Alternatively, under these

[5]Clearly, the acquisition in school of "facility with numbers," say, is not solely motivated by the possibility of taking care of number-intensive tasks in one's future household. However, what is required by our reasoning is that this is only *a part* of one's incentives in acquiring "facility with numbers."

[6]To the best of our knowledge, no analysis of mixed-strategy equilibria exists in extant models with continuous efforts, with the exception of Croson *et al.* (2006a, 2006b). However, their model has linear benefit function and linear cost. Thus, the mixed-strategy equilibria they identify are of the "bang-bang" type: either agents contribute all of their endowment or they contribute nothing. This structure is closer to the 0-1 decision framework in Harrington (2001) or Weesie (1994) than to ours.

conditions (satisfied, for instance, by benefit functions for the public good that display constant relative risk aversion), greater *ex ante* heterogeneity of the group increases agents' payoffs.

With or without private information, equilibria with two or more active players are inefficient for two reasons: wasteful duplication of efforts and less-than-ideal "best-shot" efforts. We present mechanisms that mitigate these inefficiencies. However, these mechanisms require coordination, and, under private information, commitment and side payments.

The remainder of the paper proceeds as follows. Section 2 lays out the elements of the best-shot public good game common to the full-information and private-information scenarios. Analysis of the full-information model is in Section 3, while the private-information model is found in Section 4. Section 5 provides mechanisms that improve efficiency in the provision of the best-shot public good. Technical proofs are in the Appendix.

2 The model: common elements

We suppose there are $n \geq 2$ group members, indexed by $i = 1, \ldots, n$, any of whom may exert effort on behalf of the group. Player i's marginal cost of effort is given by c_i, and players have a common benefit function $v(G)$ for the public good, where we assume $v' > 0$ and $v'' < 0$. Players can choose any nonnegative level of effort. If players choose efforts $x_1, \ldots, x_n$, then the realized level of the public good is $G = \max\{x_1, \ldots, x_n\}$, as specified by the "best shot" production function of Hirshleifer (1983). Thus, player i's realized utility is $v(G) - c_i x_i$.

3 Full information about costs

In this section we suppose players' marginal costs of effort are common knowledge. We first consider symmetric players, later allowing asymmetry.

3.1 Symmetric games

Suppose $c_i = c > 0$ for all i and $v'(0) > c$; the second inequality ensures that in equilibrium the good is provided with positive probability. Even in this symmetric game, asymmetric equilibria exist. One equilibrium has player 1 exert his "stand alone" effort $x^{\text{sa}}(c) \equiv \arg\max_g \{v(g) - cg\}$ while all others exert no effort. (In this section c is common and known, so we abbreviate $x^{\text{sa}}(c)$ with x^{sa}.) We next investigate equilibria in which more than one player is active—an *active* player exerts positive effort with positive probability; an *inactive* player always gives zero effort. We (first) investigate equilibria in which the active players follow the same strategy. We label such an equilibrium "semi-symmetric." Let m denote the number of active players in equilibrium. As already described, if $m = 1$, the (essentially) unique equilibrium has the active player choose effort x^{sa} and all others choose effort 0.

Now suppose the number of active players is $m \geq 2$. In this case, pure-strategy semi-symmetric equilibria do not exist. So, searching for a semi-symmetric mixed-strategy equilibrium, we let F denote the cumulative distribution function (cdf) for effort by each of the active players. Let $\underline{x} \equiv \inf\{x \,|\, F(x) > 0\}$ and $\bar{x} \equiv \sup\{x \,|\, F(x) < 1\}$ denote the lower and upper limits of the support of F. If F is the common cdf for effort, then $(F(\cdot))^{m-1}$ is the cdf of the maximum effort of $m - 1$ active players, and an active player's expected payoff from effort x is

$$\begin{aligned} V(x) &= -cx + v(x)(F(x))^{m-1} + \int_x^{\bar{x}} v(y)\, d\big((F(y))^{m-1}\big) \\ &= -cx + v(\bar{x}) - \int_x^{\bar{x}} (F(y))^{m-1} v'(y)\, dy, \end{aligned} \tag{1}$$

after integration by parts.[7]

Note that V is continuous on $[0, \bar{x}]$. Moreover, it must be that $\underline{x} = 0$ and $\bar{x} = x^{\text{sa}}$,[8] and the following lemma, proven in the Appendix, shows there are no "gaps" in the support of F.

Lemma 1 (Interval support)**.** *If F is the cdf used by active players in a semi-symmetric equilibrium having at least 2 active players, then the support of F is the interval $[\underline{x}, \bar{x}] = [0, x^{sa}]$.*

Let v^* denote the equilibrium payoff to one of the m active player's using cdf F. Then it must be that $V(x) = v^*$ on a dense subset of $[0, \bar{x}]$; and because V is continuous, it follows that V is constant on $[0, \bar{x}]$. Hence, V is differentiable on $(0, \bar{x})$, with $V'(x) \equiv 0$. Even though we have not yet established that F is continuous, it is the case that F is integrable, so

$$0 = V'(x) = -c + v'(x)(F(x))^{m-1},$$

on $(0, \bar{x})$, except for a set of Lebesgue measure 0.[9] Therefore, $(F(x))^{m-1} = c/v'(x)$ almost everywhere (a.e.)

[7] Standard versions of Integration by Parts would here require F to be differentiable, but we have not yet even shown it is continuous. The version in Billingsley (1995, Theorem 18.4) covers our situation.

[8] Suppose instead that $\underline{x} > 0$. Because $m \geq 2$, an active player is sure someone else is exerting at least effort $\underline{x}$, so the first player would prefer exerting zero effort to any effort in the interval $[\underline{x}, \underline{x} + \varepsilon)$, for $\varepsilon > 0$ sufficiently small, contradicting the assumption that the cdf F was part of an equilibrium. Next suppose instead that $\bar{x} \neq x^{\text{sa}}$. Any effort exceeding x^{sa} is strictly dominated by x^{sa}. Therefore, it must be that $\bar{x} \leq x^{\text{sa}}$. If $\bar{x} < x^{\text{sa}}$, then in equilibrium any active player receives payoff $v(\bar{x}) - c\bar{x}$, which is necessarily less than $v(x^{\text{sa}}) - cx^{\text{sa}}$. In this case, an active player would raise her payoff by exerting effort x^{sa} with probability 1, contradicting the assumption that the cdf F was part of an equilibrium.

[9] See Billingsley (1995, Theorem 31.3).

on $(0, \bar{x})$. From this expression, right-continuity of F, and continuity of v', we can identify F on all of $[0, \bar{x})$:

$$F(x) = \left(\frac{c}{v'(x)}\right)^{\frac{1}{m-1}} \quad \text{on } [0, \bar{x}). \tag{2}$$

And because $v'(\bar{x}) = c$, it follows that F is continuous at $\bar{x}$. Indeed, F is continuous on $[0, \bar{x}]$.

We next show this configuration ($m \geq 2$ active players using cdf (2) and $n - m$ inactive players) is an equilibrium to the public good game. For all players, efforts greater than $\bar{x}$ are strictly dominated by effort $\bar{x}$. Active players will not be better off becoming inactive—0 is in the support of their cdfs and $V(x)$ is continuous, so they have no profitable deviation. It only remains to show that inactive players cannot do better by becoming active. The cdf of the maximum effort of m players using the above cdf is $(F)^m$. An inactive player "deviating" to effort $x \in (0, \bar{x}]$ would have expected payoff equal to

$$\begin{aligned} V_D(x) &= -cx + v(x)(F(x))^m + \int_x^{\bar{x}} v(y)\, d((F(y))^m) \\ &= -cx + v(\bar{x}) - \int_x^{\bar{x}} (F(y))^m v'(y)\, dy \\ &= -cx + v(\bar{x}) - \int_x^{\bar{x}} \left(\frac{c}{v'(y)}\right)^{\frac{m}{m-1}} v'(y)\, dy. \end{aligned}$$

Differentiating V_D we see, for any $x \in (0, \bar{x})$,

$$\begin{aligned} V_D'(x) &= -c + c^{\frac{m}{m-1}} \left(\frac{1}{v'(x)}\right)^{\frac{1}{m-1}} \\ &< -c + c^{\frac{m}{m-1}} \left(\frac{1}{c}\right)^{\frac{1}{m-1}} && (v'(x) > c \ \ \forall\, x < \bar{x}) \\ &= 0. \end{aligned}$$

Therefore, the inactive player's payoff strictly decreases as he increases effort above zero, showing he has no profitable deviation. We have established the following.

Proposition 1 (Equilibrium characterization)**.** *There is an equilibrium to the n-player full-information game where $m \geq 2$ players use strategy F given by*

$$F(x) = \left(\frac{c}{v'(x)}\right)^{\frac{1}{m-1}} \quad \textit{on } [0, x^{sa}], \tag{3}$$

and $n - m$ players are inactive. Given $m \geq 2$ active players, if active players act symmetrically, then the equilibrium is unique up to permutations of the identities of active and inactive players.[10]

[10]There also exist equilibria in which the active players behave asymmetrically, as shown by Example 3, *infra.*

Note that the equilibrium cdf places no atoms on efforts above 0, but if $v'(0) < \infty$ it *does* place an atom on zero effort. Thus, even with continuous levels of efforts possible, the realized level of the public good may be zero. Also, as one expects, if the marginal cost of effort rises, less is provided, here in the sense of first-order stochastic dominance.

As the number of active players increases, each active player stochastically reduces his effort. Furthermore, this reduction in effort is sufficiently strong that even as the greater number of active players, m, increases, the distribution of the realized level of the public good, $(F(x))^m = \left(\frac{c}{v'(x)}\right)^{\frac{m}{m-1}}$, shifts leftward (in the sense of first-order stochastic dominance), with limiting cdf $c/v'(x)$.

Turning now to the relationship between group size and a player's utility, note that in every semi-symmetric equilibrium active players receive payoff $v(\bar{x}) - c\bar{x}$, while inactive players receive a payoff greater than this—all players enjoy the same distribution of the public good and inactive players are better off as they incur no costs. Therefore, comparing these equilibria, active players are indifferent among them, and inactive players prefer those having fewer active players, with just one active player being best. Without regard to players' identities, these equilibria are thus Pareto ranked, with better equilibria having fewer active players.

Related results on the connection between group size and full-information best-shot (or "better-shot") public good provision appear in Harrington (2001) and in Cornes and Hartley (2007). Harrington (2001) considers a binary model of effort ("help" or "ignore") in which the group is successful if at least one agent exerts effort ("helps"). In contrast, our technology allows gradation of efforts. Cornes and Hartley (2007) consider a CES production function for the public good and change the elasticity of substitution parameter so that the production function approaches the best-shot one (a situation they label a "better-shot" production function). They focus on the level of public good provision in pure-strategy equilibria. In contrast, we focus on equilibria of the "best-shot" production function, which for symmetric equilibria necessarily involve mixed strategies.

The following examples illustrate the symmetric equilibrium in the best-shot public good game.

Example 1. *$m = 2$ and $v(x) = x - \frac{1}{2}x^2$.*

Here the equilibrium cdf is $F(x) = \frac{c}{1-x}$, so with probability c a player exerts no effort and the maximum level of effort is $1 - c$. Also, the probability density function (pdf) of effort on $(0, 1-c]$ is $f(x) = \frac{c}{(1-x)^2}$, which is increasing. So, conditional on effort being exerted, greater levels are *increasingly* likely. □

Example 2. *$m = 2$ and $v(x) = 2\sqrt{x}$.*

Here we see $F(x) = c\sqrt{x}$. There are no atoms in this distribution of effort, and the maximum level of effort is $\frac{1}{c^2}$. Also the pdf of effort is $f(x) = \frac{c}{2\sqrt{x}}$, which is decreasing. So greater levels are *decreasingly* likely. □

The difference between the two previous examples relates to the underlying risk aversion exhibited by the benefit function $v(\cdot)$. To see this, observe that on $(0, \bar{x})$, differentiation of (2) yields

$$\begin{aligned} f(x) &= \frac{1}{m-1}\left(\frac{c}{v'(x)}\right)^{\frac{1}{m-1}-1}\left(-\frac{c}{(v'(x))^2}v''(x)\right) \\ &= \frac{1}{m-1}\left(\frac{c}{v'(x)}\right)^{\frac{1}{m-1}}\left(-\frac{v''(x)}{v'(x)}\right) \\ &= \frac{1}{m-1}F(x)r_A(x), \end{aligned} \tag{4}$$

where $r_A(x) \equiv -v''(x)/v'(x)$ denotes the Arrow-Pratt coefficient of absolute risk aversion. Thus, if risk aversion is increasing in the level of the public good (as in the first example), then the density is sure to increase with x; if risk aversion decreases sufficiently quickly, then the density decreases with x, as in the second example. Rewriting (4) as

$$\frac{f(x)}{F(x)} = \frac{r_A(x)}{m-1}, \tag{5}$$

we obtain the following:

Remark 1. *Suppose x^{sa} solves $v'(x) = c$. Consider an alternative concave benefit function $w(g)$ for which $w'(x^{sa}) = v'(x^{sa})$. Let F_v and F_w denote the semi-symmetric equilibrium cdfs with $m \geq 2$ active players, where the benefit function is either v or w, respectively. Then F_v and F_w have the same supports, $[0, x^{sa}]$. If w exhibits greater (absolute) risk aversion than does v on this support, then, by (5), F_w reverse hazard rate dominates F_v; as a consequence, F_w FOSD F_v.*

One interpretation of the previous remark is that risk aversion promotes efforts—while holding constant a players' stand-alone effort level, an increase in risk aversion causes players to exert greater effort in the sense of first-order stochastic dominance.

An intuitive way to understand the role of risk aversion in Remark 1 is to view (active) agent i's decision to exert effort x as buying insurance against a low realization of other agents' best shot. In particular, one may think that agent i owns a stock that pays out just as the best shot of all other agents (i.e., according to the distribution $[F(x)]^{m-1}$); exerting effort x is equivalent to buying a put option on this stock with strike price x at total cost cx. Consider now two benefit functions, w and v, and let the absolute risk aversion of w be larger than that of v. By the logic of mixed-strategy equilibrium, agent i is indifferent between buying no insurance (i.e., contribute 0) and buying the put option at price cx, both under w and under v. At the same time agent i values insurance more under w than under v, ceteris paribus. Therefore, agent i must own different stocks under w and under v. In particular, under w the stock must be "better," in line with the results in Remark 1.

3.2 Asymmetric games

Next we extend the situation described above by allowing players' costs to differ. For simplicity, assume there are two players and $0 < c_1 < c_2$. Define $x_i^{\mathrm{sa}} = x^{\mathrm{sa}}(c_i)$, $i = 1, 2$, and note that $x_1^{\mathrm{sa}} > x_2^{\mathrm{sa}}$. If player i alone were be active, he would exert effort x_i^{sa}. So, clearly, one equilibrium has player 1 exert effort x_1^{sa} while player 2 exerts none. Alternatively, if player 2's cost is not much higher than c_1, then player 1 would be better off exerting no effort when player 2 exerts effort x_2^{sa} than he (player 1) would be when choosing x_1^{sa}, that is,

$$v(x_2^{\mathrm{sa}}) \geq v(x_1^{\mathrm{sa}}) - c_1 x_1^{\mathrm{sa}}. \tag{6}$$

Consequently, in some cases we will find a second equilibrium in which the player with lower cost completely free rides on the less able player.[11] There are no other pure strategy equilibria.

From the earlier discussion of symmetric games, one might expect to find an equilibrium where both players randomize over an interval $[\underline{x}, \bar{x}]$. However, given $c_1 < c_2$, no such equilibrium exists. Suppose, to the contrary, such an equilibrium exists. The largest effort player 2 would conceivably take is x_2^{sa}. If both players were active, player 1 choosing $x_1 = \bar{x} \leq x_2^{\mathrm{sa}}$ would earn payoff less than or equal to $v(x_2^{\mathrm{sa}}) - c_1 x_2^{\mathrm{sa}}$, which is strictly less than when choosing x_1^{sa}. Therefore, the proposed strategies cannot form an equilibrium. It turns out, however, that there may exist equilibria in which players' strategies have *different* supports, as shown in the following example. Moreover, there exists an equilibrium that displays this property: the closer c_1 and c_2, the smaller the difference between players' equilibrium strategies and the semi-symmetric equilibrium we identified in Section 3.1. Thus, the heretofore unexplored class of equilibria we described earlier is in this sense robust to small, asymmetric changes in effort costs.

Example 3. *Suppose there are two players and $v(x) = 2\sqrt{x}$.*

Here we exhibit three types of equilibria. In the first, effort is exerted by only one player, and this need not be the low-cost player. In the second, both players take effort, one using a pure strategy and the other randomizing. Finally, we provide an equilibrium in which each player randomizes over $k \geq 2$ effort levels, none of which are common between the two players.

[11] In a full-information model, Chowdhury *et al.* (2011) find a similar equilibrium in a contest among teams where each team's effective effort is given by the greatest effort of any member of the team. A team's probability of winning is determined through a standard contest success function having as arguments each team's best effort. In some (pure strategy) equilibria the most able player on a team exerts no effort.

In the first equilibrium only player 2 exerts effort. In such an equilibrium, player 2 takes effort $x_2^{\rm sa}$ and player 1 gives zero effort. This is surely an equilibrium if $c_1 \geq c_2$. Even if $c_1 < c_2$, so $x_1^{\rm sa} > x_2^{\rm sa}$, it can remain an equilibrium if (6) is satisfied, which, for $v(x) = 2\sqrt{x}$, becomes

$$2\sqrt{1/c_2^2} \geq 2\sqrt{1/c_1^2} - c_1/c_1^2 = 1/c_1,$$

or $c_2 \leq 2c_1$. Thus, it is an equilibrium for player 1 to exert no effort and player 2 to exert effort $x_2^{\rm sa}$ with probability 1 if and only if $c_2 \leq 2c_1$.

Next consider a hybrid equilibrium where player 2 chooses some level x_2^* with probability 1 and player 1 randomizes between 0 and x_1^*. Such strategies can form an equilibrium only if $0 < x_2^* < x_1^*$. Now, if player 1 exerts effort greater than x_2^*, then it must be that $x_1^* = x_1^{\rm sa}$. If player 1 exerts no effort, his payoff is $v(x_2^*)$; and if he exerts effort $x_1^{\rm sa}$, his payoff is $1/c_1$. Therefore, for player 1 to be willing to randomize between 0 and $x_1^{\rm sa}$, it must be that

$$\frac{1}{c_1} = v(x_1^{\rm sa}) - c_1 x_1^{\rm sa} = v(x_2^*) = 2\sqrt{x_2^*},$$

implying $x_2^* = 1/(4c_1^2)$, which, indeed, is less than $x_1^{\rm sa} = 1/c_1^2$. Now, given player 1 exerts effort $x_1^{\rm sa}$ with probability p, player 2's payoff from effort $x < x_1^{\rm sa}$ is $(1-p)v(x) + pv(x_1^{\rm sa}) - c_2 x$, which is maximized with respect to x when $c_2 = (1-p)v'(x) = (1-p)/\sqrt{x}$. In equilibrium, this first-order condition must hold where $x = x_2^* = 1/(4c_1^2)$, so

$$c_2 = \frac{1-p}{\sqrt{x_2^*}} = 2c_1(1-p),$$

yielding $p^* = 1 - \frac{c_2}{2c_1} = (2c_1 - c_2)/(2c_1)$, with a corresponding payoff of $(8c_1 - 3c_2)/(4c_1^2)$ for player 2. Additionally, equilibrium requires that player 2 not have a profitable deviation to $x_2^{\rm sa}$, a requirement that is satisfied if

$$\frac{1}{c_2} = v(x_2^{\rm sa}) - c_2 x_2^{\rm sa} \leq \frac{8c_1 - 3c_2}{4c_1^2},$$

which in turn is satisfied if and only if $2c_1/3 \leq c_2 \leq 2c_1$. Finally, because $p^* \in (0,1)$ if and only if $0 < c_2 < 2c_1$, we see this hybrid equilibrium with player 1 randomizing and player 2 exerting a certain effort exists if and only if $2c_1/3 \leq c_2 < 2c_1$.

Beyond the equilibria just discussed, a wide variety of equilibria exist in which both players strictly randomize over finite sets of efforts. We will illustrate the situation with a k-step mixed-strategy equilibrium, where $k \geq 2$. The support for player 1's mixed strategy is $\{x_{1,1}, \ldots, x_{1,k'}, \ldots, x_{1,k}\}$; for player 2's, $\{x_{2,1}, \ldots, x_{2,k'}, \ldots, x_{2,k}\}$. Let $p_{j,k'}$ be the probability assigned by the mixed strategy to $x_{j,k'}$, $k' = 1, \ldots, k$, and $j = 1, 2$. In the Appendix we demonstrate that if $\left(\frac{2k}{2k-1}\right) c_2 \geq c_1 > \left(\frac{2k-2}{2k-1}\right) c_2$, then the following constitutes an equilibrium:

- $x_{1,k'} = \left(\frac{2k'-1}{2k-1}\frac{1}{c_1}\right)^2$, for $k' \in 1, 2, \ldots, k$;
- $p_{1,k'} = \frac{2}{2k-1}\frac{c_2}{c_1}$, for $k' \in 1, 2, \ldots, k-1$, and $p_{1,k} = 1 - \frac{2(k-1)}{2k-1}\frac{c_2}{c_1}$;
- $x_{2,k'} = \left(\frac{2k'-2}{2k-1}\frac{1}{c_1}\right)^2$, for $k' \in 1, 2, \ldots, k$;
- and $p_{2,1} = \frac{1}{2k-1}$, and $p_{2,k'} = \frac{2}{2k-1}$, for $k' \in 2, \ldots, k$.

Note how the supports of the two players' strategies "interlace":[12]

$$0 = x_{2,1} < x_{1,1} < x_{2,2} < x_{1,2} < \ldots < x_{2,k'} < x_{1,k'} < x_{2,k'+1} < \ldots < x_{2,k} < x_{1,k} = x_1^{\text{sa}}.$$

Moreover, the larger the number of steps (k), the closer c_2 and c_1 must be for the k-step equilibrium to exist, since the interval for admissible c_1/c_2, namely, $\left(\frac{2k-2}{2k-1}, \frac{2k}{2k-1}\right]$, shrinks to the point 1 as $k \to \infty$. Also, note that if $c_1 = c_2 = c$, then, as $k \to \infty$, the above described equilibrium converges to the symmetric one in Example 2, for which $F(x) = c\sqrt{x}$. Indeed, in the step-function equilibrium, as k grows large the amount x can be approximated with player 1's k''-th level, where

$$\left(\frac{2k''-1}{2k-1} \times \frac{1}{c}\right)^2 = x \quad \text{or} \quad k'' = \frac{(2k-1)c\sqrt{x}+1}{2},$$

and for simplicity we ignore integer constraints. Therefore, we obtain

$$\Pr(x_1 \leq x) \approx \sum_{j=1}^{k''} p_{2,j} = \frac{(2k-1)c\sqrt{x}+1}{2}\frac{2}{2k-1},$$

which converges to $c\sqrt{x}$ as $k \to \infty$, as we wanted to show. This last result may be interpreted as lending robustness to our semi-symmetric equilibrium in Proposition 1, even if c_1 and c_2 are not exactly equal. □

4 Private information about costs

Here we suppose players' marginal costs of effort are private information. We first consider symmetric players; later we allow for asymmetry.

[12] To see why this interlacing arises, suppose, for example, consider consecutive two elements of player 2's support $x_{2,k'-1}$ and $x_{2,k'}$. All $x_1 \in [x_{2,k'-1}, x_{2,k'}]$ have the same chance of being the "best shot," so player 1's expected utility over this interval is strictly concave. Therefore, player 1 has at most one optimum within this interval. Moreover, there must be at least one, for otherwise $x_{2,k'-1}$ and $x_{2,k'}$ have the same probability of being the best shot, and, again by strict concavity of v both cannot be optimal for player 2.

4.1 Symmetric games

In this subsection we assume players' marginal costs of effort are independently and identically distributed random variables with cdf F on $[\underline{c}, \bar{c}]$, where $\underline{c} > 0$. We assume F has pdf f. As before, the common benefit function $v(G)$ satisfies $v' > 0$ and $v'' < 0$, and we now assume $v'(0) > \underline{c}$. Also as before, the level of the public good is $G = \max\{x_1, \ldots, x_n\}$, where x_i denotes player i's effort.

We next characterize the symmetric equilibrium, in which all players are active.[13] Let g denote the common equilibrium strategy. Standard arguments establish g is weakly decreasing. The greatest effort is provided by the player with the lowest cost. Let F^M denote the cdf of the *minimum* cost of $n-1$ draws from F: $F^M(c) = 1 - [1 - F(c)]^{n-1}$; let f^M denote the associated pdf. Thus, a player sees F^M as the cdf of the least cost (best shot) among the other players. A player with cost c acting as if he has cost c' has payoff

$$V(c', c) = -cg(c') + (1 - F^M(c'))v(g(c')) + \int_{\underline{c}}^{c'} v(g(y)) f^M(y)\, dy. \tag{7}$$

Formula (7) is understood by recognizing that player 1, acting as if his cost is c', will provide the greatest group effort if everyone else's cost exceeds c', which happens with probability $1 - F^M(c')$; and when another player has the least cost, say y, then the realized level of the public good is $g(y)$. The relevant first-order condition for optimization is

$$\begin{aligned} 0 = \left.\frac{\partial V(c', c)}{\partial c'}\right|_{c'=c} &= \left[-cg'(c') + (1 - F^M(c'))v'(g(c'))g'(c')\right|_{c'=c} \\ &= g'(c)\left[-c + v'(g(c))(1 - F(c))^{n-1}\right]. \end{aligned} \tag{8}$$

Condition (8) simply balances marginal cost and expected marginal benefit of effort: c is the marginal cost of more effort; a player exerting effort $g(c')$ obtains marginal benefit $v'(g(c))$ of additional effort, but only when he is the best-shot, and this happens with probability $(1 - F(c))^{n-1}$. Where g is strictly decreasing, we see from (8) that

$$v'(g(c)) = \frac{c}{[1 - F(c)]^{n-1}}. \tag{9}$$

Note that the right-hand side of (9) increases in c, so, because v' is decreasing, g is indeed decreasing wherever it is positive. Now define c^* as

$$c^* = \begin{cases} \bar{c} & \text{if } v'(0) = +\infty \\ \text{the unique solution of } v'(0) = \frac{c}{[1-F(c)]^{n-1}} & \text{if } v'(0) < \infty. \end{cases} \tag{10}$$

[13] There may exist semi-symmetric equilibria in which some players exert zero effort for all values of c. Whether such equilibria exist depends on a complex relationship between the strategy of the active players and the fine details of F.

Thus, in a group of n players, types exceeding c^* totally free ride, providing no effort themselves. We use (9) and (10) to characterize equilibrium strategies.

Proposition 2 (Characterization)**.** *The unique symmetric equilibrium strategy is*

$$g(c) = \begin{cases} (v')^{-1}\left(\frac{c}{[1-F(c)]^{n-1}}\right) & \text{if } c \le c^* \\ 0 & \text{if } c > c^*, \end{cases} \tag{11}$$

where c^* is given by (10)*.*

We leave a complete proof of the above proposition to the Appendix. Here, we illustrate equilibrium with an example.

Example 4. *$n = 2$ and c is uniformly distributed on $[\underline{c}, \bar{c}]$, where $0 < \underline{c} < \bar{c}$.*

Here $F(c) = (c - \underline{c})/(\bar{c} - \underline{c})$ for $c \in [\underline{c}, \bar{c}]$. First suppose $v(x) = 2\sqrt{x}$. Then we expect g to be everywhere decreasing, and indeed from (9) we find

$$g(c) = \left[\frac{(\bar{c} - c)}{(\bar{c} - \underline{c})c}\right]^2.$$

If instead $v(x) = x - \frac{1}{2}x^2$ and $\bar{c} \le 1$, then we see

$$g(c) = \begin{cases} 1 - (\bar{c} - \underline{c})\left(\frac{c}{\bar{c}-c}\right) & \text{if } c \le \frac{\bar{c}}{1+\bar{c}-\underline{c}} \\ 0 & \text{if } c > \frac{\bar{c}}{1+\bar{c}-\underline{c}}. \end{cases}$$

□

We next establish properties of equilibrium. Before proceeding further, note that, if we define through (7) the equilibrium utility of type c as $V^*(c) \equiv V(c, c)$, then by the usual incentive compatibility arguments we obtain

$$\frac{dV^*(c)}{dc} = -g(c),$$

leading to

$$V^*(c) = V^*(\underline{c}) - \int_{\underline{c}}^{c} g(y)\, dy. \tag{12}$$

Since $V^*(\underline{c})$ does not depend on either the number of players or the distribution of costs (provided the minimum of the support does not change), the behavior of $\int_{\underline{c}}^{c} g(y)\, dy$ is of particular interest in understanding changes in interim utility.

We begin our analysis with the effects of an increase in the number of players. For given c, if n increases then the right-hand side of (9) becomes larger, so $g(c)$ must fall to preserve the equality. Moreover, the cost c^* above which a player becomes inactive decreases with n. So, individual efforts decrease pointwise with n. Therefore, as shown by (12), as the number of players increases, interim utility of the existing types increases. This is true for all types but $\underline{c}$, and in particular for inactive *types*—those types that in equilibrium do not take effort, i.e., for $c \geq c^*$.

We next turn our attention to the consequences of changes in the distribution of costs. It is immediate to see that if cost increases (F shifts rightward) in the sense of first-order stochastic dominance (FOSD) while retaining the same $\underline{x}$, then, for any c, the right-hand side of (9) becomes smaller, so $g(c)$ increases. Through (12) this cost increase results in a decrease of interim utilities. Moreover, since V^* is decreasing in c, *ex ante* expected utility decreases as well.

The analysis of the consequences of second-order stochastic dominance (SOSD) shifts is more complex. Nonetheless, under certain conditions on v, it is still possible to ascertain the direction of changes in interim and *ex ante* utility. For the rest of this section, we consider two continuous distributions of costs over $[\underline{c}, \bar{c}]$, F_1 and F_2, such that F_2 second-order stochastically dominates F_1: $\int_{\underline{c}}^{y} (F_1(c) - F_2(c))\, dc \geq 0$ for all y. For simplicity, we assume F_1 and F_2 have only finitely many points of strict intersection. We now denote now the equilibrium effort function under distribution F_i as $g\,(c|F_i)$, $i = 1, 2$. As before, these equilibrium effort functions are determined through (9), which may be rewritten as

$$v'(g(c\,|\,F_i)) = \frac{c}{1 - F_i^M(c)},$$

where F_i^M is the cdf of the minimum of $n-1$ draws from cdf F_i. The previous equation leads us to implicitly define $\psi\,(c, z)$ through the equation $v'(\psi(c, z)) = c/(1 - z)$. We then obtain the following preliminary result relating changes in the distribution of cost to changes in the equilibrium strategy.

Lemma 2. *If F_2 SOSD F_1 and $\frac{\partial^2 \psi}{\partial z^2} \geq 0$, then $\int_{\underline{c}}^{y} (g(c|F_2) - g(c|F_1))\; dc \geq 0$.*

Letting $V_i^*(c)$ denote the equilibrium interim utility of type c when the cost distribution is F_i, $i = 1, 2$, we see from (12) that[14]

$$V_1^*(c) - V_2^*(c) = \int_{\underline{c}}^{c} g(y|F_2)\, dy - \int_{\underline{c}}^{c} g(y|F_1)\, dy,$$

so Lemma 2 implies that the interim utility of type c is larger under F_1 than under F_2. As for *ex ante* utility,

[14]Because a player with value $\underline{c}$ is sure to be the "best shot," $V_1(\underline{c}) = V_2(\underline{c})$.

we have

$$\begin{aligned}\int_{\underline{c}}^{\bar{c}} V_1^*(c) f_1(c)\,dc - \int_{\underline{c}}^{\bar{c}} V_2^*(c) f_2(c)\,dc &= \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{c} g(y|F_2)\,dy\, f_2(c)\,dc - \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{c} g(y|F_1)\,dy\, f_1(c)\,dc \\ &\geq \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{c} g(y|F_2)\,dy\, f_1(c)\,dc - \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{c} g(y|F_1)\,dy\, f_1(c)\,dc \\ &= \int_{\underline{c}}^{\bar{c}} \left[\int_{\underline{c}}^{c} g(y|F_2) dy - g(y|F_1)\,dy \right] f_1(c)\,dc \\ &\geq 0, \end{aligned}$$

where the first inequality follows because $\int_{\underline{c}}^{c} g(y|F_2)\,dy$ is an increasing concave function of c and F_2 SOSD F_1, while the second inequality follows from Lemma 2. This discussion proves the following comparison.

Proposition 3 (SOSD utility comparisons)**.** *Under the condition of Lemma 2, both* ex ante *and interim payoffs are greater under the "more risky" distribution of costs.*

Intuitively, Proposition 3 is a consequence of the convexity in the best-shot production function. A natural question is then whether the sufficient condition in Lemma 2 is satisfied for common utility functions. In the proof of Lemma 2 (see the Appendix), equation (26) demonstrates that

$$\frac{\partial^2 \psi(c,z)}{\partial z^2} \geq 0 \iff \frac{v'(\psi(c,z))}{v''(\psi(c,z))} \frac{v'''(\psi(c,z))}{v''(\psi(c,z))} \geq 2; \tag{13}$$

hence, the underlying risk aversion exhibited by the benefit function $v(\cdot)$ is again crucial, as it was for the game without private information. The right-hand condition in (13) has several implications. First, because v is strictly concave, we see that the inequality in (13) is satisfied only if $v''' > 0$, that is, marginal utility is convex. Indeed, inequality (13) reduces to the condition that $r_A(x)$ decrease "sufficiently" fast.[15] This is the case, for instance, for the constant-relative-risk-aversion function $v\,(x) = x^a$, $0 < a < 1$, where

$$\frac{v'}{v''} \frac{v'''}{v''} = \frac{2-a}{1-a} \geq 2.$$

4.2 Asymmetric games

Describing equilibrium behavior among asymmetric players is considerably more complicated than when they are symmetric, so we restrict our analysis to the case of two players. In this section, we analyze

[15] Recalling that the Arrow-Pratt measure of risk aversion is given by $r_A(x) = -v''(x)/v'(x)$, we see

$$\frac{dr_A(x)}{dx} = -(r_A(x))^2 \left(\frac{v'(x)v'''(x)}{(v''(x))^2} - 1 \right).$$

For r_A to be decreasing it must be that the fraction on the right-hand side of (13) exceeds 1. Condition (13) requires this fraction to exceed 2, thus requiring r_A to decrease even faster.

the two-player, independent-cost game in which players' marginal costs of effort are drawn from different distributions. In particular, we let c_i be distributed on $[\underline{c}, \bar{c}]$ according to F_i, with continuous density f_i, for $i = 1, 2$.

We look for equilibrium strategies g_1 and g_2. Our characterization begins for effort levels γ at which both g_1 and g_2 are strictly decreasing, so that they admit inverse functions ϕ_1 and ϕ_2. If player 1 with cost c_1 takes effort γ, then his payoff is

$$V_1(\gamma, c_1) = -c_1\gamma + (1 - F_2(\phi_2(\gamma)))v(\gamma) + \int_{\underline{c}}^{\phi_2(\gamma)} v(g(y))f_2(y)\,dy, \tag{14}$$

and the first-order condition with respect to γ yields

$$c_1 = v'(\gamma)(1 - F_2(\phi_2(\gamma))).$$

Proceeding similarly for player 2, and noting that for $i = 1, 2$ $c_i = \phi_i(\gamma)$, we obtain the following system of two equations in the two unknowns $\phi_1(\gamma)$ and $\phi_2(\gamma)$:

$$\begin{cases} \phi_1(\gamma) = v'(\gamma)(1 - F_2(\phi_2(\gamma))) \\ \phi_2(\gamma) = v'(\gamma)(1 - F_1(\phi_1(\gamma))). \end{cases} \tag{15}$$

The above system is analogous to condition (9) for the symmetric case. However, further complications arise when the system (15) fails to produce two functions $\phi_1(\gamma)$ and $\phi_2(\gamma)$ that are strictly decreasing throughout. The following example illustrates the adjustments that are necessary to identify equilibrium in such a circumstance.

Example 5. $F_1(c_1) = \frac{10}{9}\left(c_1 - \frac{1}{10}\right)$ *and* $F_2(c_2) = \frac{100}{99}\left((c_2)^2 - (\frac{1}{10})^2\right)$ *on* $[\frac{1}{10}, 1]$, *with* $v(x) = 2\sqrt{x}$.

Player 2 has stochastically higher costs than player 1, so we might expect player 2 will be more likely to free ride. Note that $1 - F_1(c_1) = \frac{10}{9}(1 - c_1)$ and $1 - F_2(c_2) = \frac{100}{99}\left(1 - (c_2)^2\right)$. Solving the system (15) we find

$$\phi_1^H(\gamma) = \frac{20000 + 9\sqrt{\gamma}(-891\gamma + \sqrt{4000000 - 3960000\sqrt{\gamma} + 793881\gamma^2})}{20000},$$

and

$$\phi_2^H(\gamma) = \frac{891\gamma - \sqrt{4000000 - 3960000\sqrt{\gamma} + 793881\gamma^2}}{2000}.$$

Figure 1 plots ϕ_1^H and ϕ_2^H. Because equilibrium strategies must be nonincreasing in cost, these functions are potentially relevant only where they are both nonincreasing (as functions of γ). Therefore, we expect that

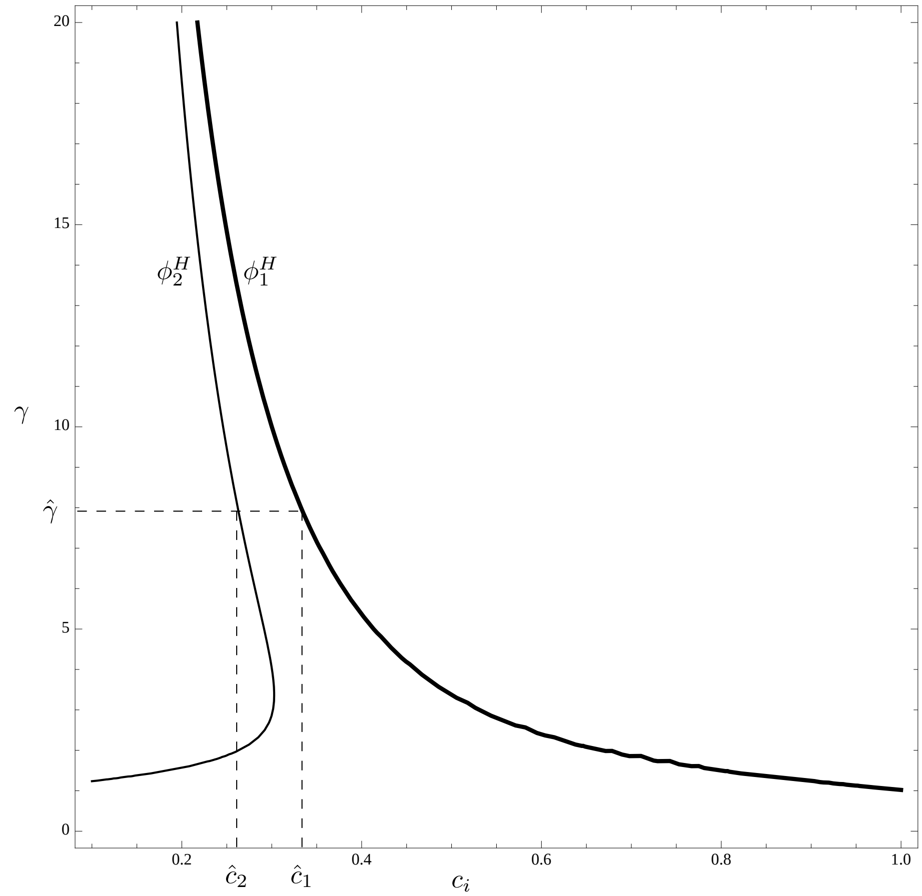

Figure 1: Possible effort functions in Example 5

for cost sufficiently high, player 2 will drop out. Now suppose $\hat{\gamma}$ is the smallest positive effort of player 2, and this happens at cost $\hat{c}_2$. From Figure 1 we see when player 1 has cost less than $\hat{c}_1$ he will face the standard problem and his optimal effort is found via (15); but with cost above $\hat{c}_1$, player 1 must consider the possibility that player 2 exerts zero effort. This leads to a reformulation of player 1's objective function when he has cost larger than $\hat{c}_1$. Now the equilibrium drop-out effort level for player 2 is found where he is just indifferent between efforts $\hat{\gamma}$ and 0, having accounted for the recalculation of player 1's strategy for costs above $\hat{c}_1$. As shown in the Appendix, this indifference requires that the equilibrium dropout level $\hat{\gamma}$ solve

$$-\phi_2^H(\hat{\gamma})\hat{\gamma} + \frac{10}{9}\left(1 - \phi_1^H(\hat{\gamma})\right) 2\sqrt{\hat{\gamma}} = -2 \times \frac{100}{99}\left(1 - \left(\phi_2^H(\hat{\gamma})\right)^2\right)\frac{10}{9}\log\left(\phi_1^H(\hat{\gamma})\right). \tag{16}$$

It may be verified numerically that $\hat{\gamma} \approx 11.21$. Also, to facilitate the description of equilibrium, let $\hat{c}_1 = \phi_1^H(\hat{\gamma}) \approx 0.28$ and let $\hat{c}_2 = \phi_2^H(\hat{\gamma}) \approx 0.24$. In the Appendix, we demonstrate that the following functions describe an equilibrium:

$$g_1(c_1) = \begin{cases} (\phi_1^H)^{-1}(c_1) & \text{if } c_1 \leq \hat{c}_1, \\ \left(\frac{99}{100} \times \frac{1-(\hat{c}_2)^2}{c_1}\right)^2 & \text{if } c_1 > \hat{c}_1; \end{cases}$$

and

$$g_2(c_2) = \begin{cases} (\phi_2^H)^{-1}(c_2) & \text{if } c_2 \leq \hat{c}_2, \\ 0 & \text{if } c_2 > \hat{c}_2, \end{cases}$$

where here $(\phi_2^H)^{-1}$ derives from the portion of ϕ_2^H where ϕ_2^H is a strictly decreasing function.

The behavior described by g_1 and g_2 above shows that the player with the worse (higher) cost distribution—player 2—drops out for sufficiently large realizations of his cost, that is, when $c_2 > \hat{c}_2$. Therefore, for effort levels smaller than $\hat{\gamma} = g_2(\hat{c}_2) = g_1(\hat{c}_1)$, type $c_1 > \hat{c}_1$ bears the full cost of provision and chooses his best effort taking into account that he will be the "best shot" if $c_2 > \hat{c}_2$, which occurs with probability $1 - F_2(\hat{c}_2) = \frac{100}{99}(1 - (\hat{c}_2)^2)$. In contrast, when $c_1 \leq \hat{c}_1$ and $c_2 \leq \hat{c}_2$, both players provide effort according to condition (15), which has solutions ϕ_1^H and ϕ_2^H. Finally, we also see that $g_2(c) < g_1(c)$ for all c, in accord with the initial expectation that the higher-cost player engages in more free riding. □

5 Mechanisms to improve efficiency

In this section we focus on symmetric games. Both in the full-information and private-information settings there are two sources of inefficiency. First, more than one player exerts effort, and, second, a player with lowest cost exerts less than his stand-alone effort, a level that would provide larger benefits to all. (Even greater overall payoffs could be achieved if the "best shot" took into account the benefits accruing to others.)

Here we provide mechanisms that remedy these two sources of inefficiency by implementing the outcome in which only one player exerts his stand-alone effort and all other players are not active. Moreover, the active player has the lowest (realized) marginal cost of effort. While we provide fairly general conditions under which these mechanisms outperform the outcomes identified in Sections 3 and 4, it is worth pointing out that these mechanisms require coordination; additionally, with private information the mechanism also requires the feasibility of side payments and the existence of an uninformed mediator with full commitment power.

5.1 Full-information games

For the symmetric full-information game described in Section 3.1, consider any semi-symmetric equilibrium with at least $m \geq 2$ active players. A mediator can raise the payoff of all players with the following mechanism. Among the m active players, the designer randomly selects one to be the sole active player, who then exerts his stand-alone effort with probability 1. Under this mechanism, the distribution of the best-shot effort shifts from having support $[0, x^{\text{sa}}]$ to being concentrated at x^{sa}; therefore, all inactive players obtain a payoff strictly greater than in the semi-symmetric equilibrium, and the sole active player obtains a payoff equal to that in the semi-symmetric equilibrium. And because a player is chosen as the active player with probability less than 1, the *ex ante* expected payoff of every player that would have been active in the original equilibrium increases. Indeed, a small transfer to the sole active player, funded by all others, can effect an *ex post* utility improvement for all players.

5.2 Private-information games

Now consider the unique symmetric equilibrium of the symmetric private-information game of Section 4.1. The random selection of a sole active player will not generally yield a Pareto improvement in interim utilities as the selected player may have a high cost of effort and the effort sufficient to render improvement for the inactive players may leave the active player worse off than in the noncooperative equilibrium. Instead, in the private-information context, the mechanism designer elicits information from players before choosing which player should be active.

The following uses all the strength of the standard mechanism design approach and agents' risk-neutrality in monetary transfers. To incorporate transfers we make the additional modeling assumption that dollars and effort enter the utility function linearly so that receiving a payment of $1 would exactly offset an increase in effort of amount $1/c$ for a player with marginal cost of effort c. *Ex ante* the designer gives to each player a fixed sum of money; as the mechanism unfolds, each player transfers to the designer a sum depending on that player's realized cost. These payments are designed so that the budget is balanced *ex ante*.

We will see that the mechanism addresses the two previously identified sources of inefficiency in the best-shot public good game. To describe the mechanism, we recall the earlier notation where $x^{\text{sa}}(c)$ denotes the stand-alone effort of a player with cost c, and F^M and f^M denote the cdf and pdf of the minimum of $n-1$ draws from the distribution F. Also, since we will contemplate changes in the number of players, to highlight the dependence of strategies on the number of players in the game, we denote by c_n^* the value in (10) and by g_n the equilibrium strategy in (11) when there are n players. Note that $g_1(\cdot) = x^{\text{sa}}(\cdot)$. The direct mechanism operates as follows:

1. Upon accepting the mechanism, each agent receives the fixed amount $E_T = \int_{\underline{c}}^{\bar{c}} T(y)\, dF(y)$, where $T(y)$ is defined next.

2. After observing their own (private) marginal costs, agents make announcements c_i^a and make transfers

$T(c_i^a)$ to the mechanism designer, where

$$T(c_i^a) \equiv \int_{\underline{c}}^{c_i^a} x^{\text{sa}}(y) y f^M(y)\, dy. \tag{17}$$

3. Let $I_{\min} \equiv \{j \in \{1, \ldots, n\} \mid c_j^a \leq c_k^a \;\; \forall k \in \{1, \ldots, n\}\}$ denote the set of players announcing the least cost. The designer chooses a player with index i', say, in $I_{\min}$ to be the sole active player, who then exerts effort in the amount $x^{\text{sa}}(c_{i'}^a)$ (all others exert no effort).

Provided this mechanism is incentive compatible, it solves the problem of identifying the low-cost player, and then that player exerts greater effort than he would in the symmetric equilibrium of the original game. The transfer function T is calibrated to ensure interim incentive compatibility. In particular, type c's utility when announcing $c^a < \bar{c}$ is

$$U(c^a, c) \equiv E_T + \int_{\underline{c}}^{c^a} v(x^{\text{sa}}(y)) f^M(y)\, dy + (1 - F^M(c^a))\big[v(x^{\text{sa}}(c^a)) - c x^{\text{sa}}(c^a)\big] - T(c^a).$$

We now have

$$\begin{aligned}
\frac{\partial U(c^a, c)}{\partial c^a} &= v(x^{\text{sa}}(c^a)) f^M(c^a) - f^M(c^a) \left(v(x^{\text{sa}}(c^a)) - c x^{\text{sa}}(c^a)\right) \\
&\qquad + (1 - F^M(c^a))(v'(x^{\text{sa}}(c^a)) - c) \frac{dx^{\text{sa}}(c^a)}{dc^a} - T'(c^a) \\
&= f^M(c^a) c x^{\text{sa}}(c^a) + (1 - F^M(c^a))(c^a - c) \frac{dx^{\text{sa}}(c^a)}{dc^a} - c^a x^{\text{sa}}(c^a) f^M(c^a) \\
&= (c - c^a) \left[x^{\text{sa}}(c^a) f^M(c^a) - (1 - F^M(c^a)) \frac{dx^{\text{sa}}(c^a)}{dc^a} \right],
\end{aligned}$$

where the second equality follows from $v'(x^{\text{sa}}(c^a)) = c^a$ and from differentiating (17). Therefore, because $dx^{\text{sa}}(y)/dy \leq 0$, U is strictly quasi-concave with respect to c^a, which guarantees incentive compatibility. Now, for a player with cost c, let $U^A(c) \equiv U(c, c)$ denote the interim utility under the alternative mechanism.

We now turn to the comparison of interim utilities. Under truth-telling, for type $c \in (\underline{c}, \bar{c})$ the difference in interim utility between this alternative mechanism and the original game is

$$\begin{aligned}
U^A(c) - V^*(c) = E_T &+ \int_{\underline{c}}^{c} \left[v(x^{\text{sa}}(y)) - y x^{\text{sa}}(y)\right] f^M(y)\, dy + (1 - F^M(c)) \left[v(x^{\text{sa}}(c)) - c x^{\text{sa}}(c)\right] \\
&- \left\{ \int_{\underline{c}}^{c} \left[v\left(g_n(y)\right) - c g_n(c)\right] f^M(y)\, dy + (1 - F^M(c)) \left[v\left(g_n(c)\right) - c g_n(c)\right] \right\} \\
> E_T &+ \int_{\underline{c}}^{c} \left[v(x^{\text{sa}}(y)) - y x^{\text{sa}}(y)\right] f^M(y)\, dy - \int_{\underline{c}}^{c} \left[v\left(g_n(y)\right) - c g_n(c)\right] f^M(y)\, dy
\end{aligned}$$

$$> E_T + \int_{\underline{c}}^{c} \underbrace{v'(x^{\text{sa}}(y))}_{=y} (x^{\text{sa}}(y) - g_n(y)) \, f^M(y) \, dy - \int_{\underline{c}}^{c} y x^{\text{sa}}(y) f^M(y) \, dy$$
$$= E_T - \int_{\underline{c}}^{c} y g_n(y) f^M(y) \, dy,$$

where the first inequality follows because the amount $x^{\text{sa}}(c)$ is the stand-alone effort of type c and the second follows because v is strictly concave. Consequently, for all $c \in (\underline{c}, \bar{c})$,

$$U^A(c) - V^*(c) > E_T - \int_{\underline{c}}^{c} y g_n(y) f^M(y) \, dy$$
$$\geq E_T - \int_{\underline{c}}^{\bar{c}} y g_n(y) f^M(y) \, dy.$$

We recalculate E_T as follows:

$$E_T = \int_{\underline{c}}^{\bar{c}} T(c) \, dF^M(c) = \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{c} x^{\text{sa}}(y) y f^M(y) \, dy \; dF^M(c)$$
$$= \int_{\underline{c}}^{\bar{c}} \int_{y}^{\bar{c}} dF^M(c) \, x^{\text{sa}}(y) y f^M(y) \, dy = \int_{\underline{c}}^{\bar{c}} y x^{\text{sa}}(y)(1 - F(y)) f^M(y) \, dy.$$

With this last expression for E_T, for any $c \in (\underline{c}, \bar{c})$ we have

$$U^A(c) - V^*(c) > \int_{\underline{c}}^{\bar{c}} y \left[x^{\text{sa}}(y)(1 - F(y)) - g_n(y) \right] f^M(y) \, dy. \tag{18}$$

Because U^A and V^* are continuous, it follows that the right-hand side of (18) is a lower bound for the utility improvement under the alternative mechanism, for *all* $c \in [\underline{c}, \bar{c}]$. The following proposition provides sufficient conditions for the integrand in (18) to be strictly positive, from which it follows that the allocation under the alternative mechanism interim Pareto dominates the unique symmetric equilibrium in the original game.

Proposition 4 (Interim inefficiency)**.** *If either*

(i) $y x^{sa}(y)$ *is decreasing in* y*; or*

(ii) $v'(0) < \infty$, $f(\underline{c}) > 0$, *and* n *is sufficiently large; or*

(iii) $v'(0) = \infty$, $f(\underline{c}) > 0$, $\bar{c} < \infty$, $\limsup_{y \to \infty} y x^{sa}(y) < \infty$, *and* n *is sufficiently large;*

then $U^A(c) > U^*(c)$ *for all* $c \in [\underline{c}, \bar{c}]$.

If we think of the public good as a private good, then $x^{\text{sa}}(c)$ would be an individual's demand when the price is c. Condition (i) simply says that as price rises, total expenditure on the good falls, so demand is elastic. This turns out to be equivalent to the coefficient of relative risk aversion for $v(\cdot)$ being less than 1 over the relevant range. This condition is satisfied, for example, by $v(x) = x^a$, where $0 < a < 1$. Condition (iii) imposes the much weaker requirement that expenditures not explode as the price becomes very large; condition (ii) can omit such a requirement because the assumption $v'(0) < \infty$ ensures that demand (i.e., effort) will be zero for sufficiently large c.

6 Conclusion

We have analyzed the private provision of a public good with "best-shot" production function. Since its first introduction by Hirshleifer (1983), the best-shot public good game has found many applications, e.g. nuclear deterrence, research and development, the creation of censorship-resistant networks, and the building of tsunami warning systems. Among these applications, we have focused on situations for which agents efforts can be graduated, that is, efforts are not restricted to being dichotomous.

Our main contribution to the best-shot public good problem is the analysis of symmetric equilibria, both with complete information and with private information. Despite the breadth of extant applications of the best shot public good problem, symmetric equilibria have received little attention in the literature, with the notable exceptions of Harrington (2001) and Xu (2001), who however operate within a dichotomous-effort framework.

We characterize all semi-symmetric equilibria—those such that agents with a positive probability of contributing toward the good follow the same strategy—when players have a commonly known marginal cost of effort. With at least two such active players, equilibrium is in mixed strategies, and we determine the relation between the equilibrium distribution of effort and the risk aversion exhibited by the benefit function for the public good: the more risk averse players are, the larger is their equilibrium contribution. Also, we show that semi-symmetric equilibria can be Pareto ranked, with better equilibria having fewer active players.

We characterize the symmetric Bayesian equilibrium when marginal costs of effort are private information. In this case, increasing the number of group members increases players' interim payoffs. We also perform comparative statics results on the distribution of marginal costs; among other results, we show that greater *ex ante* heterogeneity of the group increases agents' payoffs.

Our results are especially relevant when coordination, commitment, and side-payments are problematic.[16] Otherwise, we show simple welfare-improving mechanisms that are especially effective for large group sizes. Therefore, beyond filling a need for a more thorough analysis of an important problem in the literature, our welfare analysis provides a basis for the cost-benefit analysis of public intervention that is more complete

[16] As we argue in the Introduction, one such instance occurs if, while not yet knowing the members of their group, agents can nonetheless expend resources to acquire competences that lower the costs of providing future group-benefiting efforts.

than what emerges if one just focuses on the asymmetric equilibrium in which an agent provides the public good on his own.

Despite this contribution to welfare analysis, the focus of our paper remains eminently positive: we make no effort towards determining the best mechanism for public good provision, but we provide sensible Pareto-improving mechanisms. We leave the full normative analysis, and especially the problems generated in this context by private information, for future research. Also, while we do touch on some issues brought about by asymmetries between players, a full analysis of the effects of asymmetries is beyond the scope of this paper. Finally, we have restricted attention to one-shot interaction. The possibility of repeated play offers another avenue for welfare improvement. A full study of the best-shot public good problem in a repeated game framework is the subject of current research.

Appendix

Proof of Lemma 1. As explained in the text, the support of F has infimum $\underline{x} = 0$ and supremum $\bar{x} = x^{\text{sa}}$. It only remains to show that the support of F has no gaps. Suppose to the contrary, that there exists an interval (x^l, x^h), with $0 \leq x^l < x^h \leq \bar{x}$, in which no efforts fall, and x^l and x^h are in the support of F. Let v^* denote an active player's payoff in the semi-symmetric equilibrium where active players use cdf F. In equilibrium it must be that $V(x) = v^*$ a.e.-F, and because V is continuous it follows that $V(x) = v^*$ on the support of F. Therefore, $V(x^l) = V(x^h)$, which in turn implies

$$\begin{aligned}
0 &= V(x^h) - V(x^l) \\
&= \left(-cx^h + v(\bar{x}) - \int_{x^h}^{\bar{x}} (F(y))^{m-1} v'(y)\, dy\right) - \left(-cx^l + v(\bar{x}) - \int_{x^l}^{\bar{x}} (F(y))^{m-1} v'(y)\, dy\right) \\
&= -(x^h - x^l)c + \int_{x^l}^{x^h} (F(y))^{m-1} v'(y)\, dy \\
&< -(x^h - x^l)c + (x^h - x^l)(F(x^l))^{m-1} v'(x^l),
\end{aligned}$$

where the inequality follows because F is constant on $[x^l, x^h]$ and v' is strictly decreasing. The extremes of the previous equations yield

$$c < (F(x^l))^{m-1} v'(x^l). \tag{19}$$

For any $x \in (x^l, x^h)$ we have

$$V(x) = -cx + v(x)(F(x^l))^{m-1} + \int_{x^h}^{\bar{x}} v(y)\, d(F(y))^{m-1},$$

at which points

$$V'(x) = -c + v'(x)(F(x^l))^{m-1}. \tag{20}$$

Now

$$0 \geq \lim_{x \downarrow x^l} V'(x) = -c + v'(x^l)(F(x^l))^{m-1}, \tag{21}$$

where the inequality follows because, otherwise, since V is continuous, an effort slightly greater than x^l would yield a payoff strictly exceeding the payoff in $(x^l - \varepsilon, x^l)$, for sufficiently small $\varepsilon > 0$, contradicting the assumption that F is an equilibrium strategy. But then (21) contradicts (19). Hence, it must be that there are no gaps in the support of F. □

Proof that the k-step strategies of Example 3 form an equilibrium. To see that $x_{1,k'}$ is optimal against player 2's strategy, note first that, over the interval $(x_{2,k'}, x_{2,k'+1})$,

$$x_{1,k'} \in \operatorname*{argmax}_g \left(\sum_{j=1}^{k'} p_{2,j} \right) v(g) + \sum_{j=k'+1}^{k} v(x_{2,j}) p_{2,j} - c_1 g,$$

since, for $v(x) = 2\sqrt{x}$,

$$\begin{aligned}
c_1 &= \left(\sum_{j=1}^{k'} p_{2,j} \right) v'(g) \\
&= \frac{1}{\sqrt{g}} \left(\frac{1}{2k-1} + (k'-1)\frac{2}{2k-1} \right) \\
&= \frac{1}{\sqrt{g}} \frac{2k'-1}{2k-1}.
\end{aligned}$$

We next show that player 1's utility of taking effort $x_{1,k'}$ is the same as for effort $x_{1,k'-1}$, for $k' \in 2, \ldots, k$. It is easily verified that

$$\sum_{j=1}^{k'} p_{2,j} = \frac{1}{2k-1} + (k'-1)\frac{2}{2k-1} = \frac{2k'-1}{2k-1} = c_1 \sqrt{x_{1,k'}}.$$

Therefore, for $k' \in 2, \ldots, k$, player 1's payoff at effort $x_{1,k'}$ is

$$\begin{aligned}
\left(\sum_{j=1}^{k'} p_{2,j} \right) v(x_{1,k'}) + \sum_{j=k'+1}^{k} v(x_{2,j}) p_{2,j} - c_1 x_{1,k'} &= c_1 \sqrt{x_{1,k'}} \big(2\sqrt{x_{1,k'}} \big) + \sum_{j=k'+1}^{k} v(x_{2,j}) p_{2,j} - c_1 x_{1,k'} - c_1 x_{1,k'} \\
&= c_1 x_{1,k'} + \sum_{j=k'+1}^{k} v(x_{2,j}) p_{2,j} \\
&= c_1 x_{1,k'-1} + \sum_{j=k'}^{k} v(x_{2,j}) p_{2,j} + \big[c_1 (x_{1,k'} - x_{1,k'-1}) - v(x_{2,k'}) p_{2,k'} \big] \\
&= c_1 x_{1,k'-1} + \sum_{j=k'}^{k} v(x_{2,j}) p_{2,j} + \underbrace{\left[\frac{(2k'-1)^2 - (2k'-3)^2}{c_1 (2k-1)^2} - 2 \left(\frac{2k'-2}{2k-1} \frac{1}{c_1} \right) \frac{2}{2k-1} \right]}_{=0} \\
&= \left(\sum_{j=1}^{k'-1} p_{2,j} \right) v(x_{1,k'-1}) + \sum_{j=k'}^{k} v(x_{2,j}) p_{2,j} - c_1 x_{1,k'-1},
\end{aligned}$$

establishing the payoff equality and thus concluding the proof that $x_{1,k'}$ is optimal against player 2's strategy.

Similar calculations hold for player 2. The final consideration is whether player 2 has a profitable deviation to x_2^{sa}. This is surely not the case if $x_2^{\text{sa}} \leq x_1^{\text{sa}}$. However, if $x_2^{\text{sa}} > x_1^{\text{sa}}$ (i.e., $c_2 < c_1$), it must be verified that player 2 cannot profit from deviating to x_2^{sa}. Under the proposed equilibrium strategies, player 2's expected payoff when choosing $x_{2,1} = 0$ is

$$\sum_{k'=1}^{k} p_{1,k'} v\left(x_{1,k'}\right) = \frac{2c_1 + c_2\left(-1 + \frac{1}{(2k-1)^2}\right)}{c_1^2},$$

while the payoff from effort $x_2^{\text{sa}} = 1/c_2^2$ is $1/c_2$. Therefore, the proposed equilibrium strategies indeed form an equilibrium if

$$\begin{aligned} 0 &\leq \frac{2c_1 + c_2\left(-1 + \frac{1}{(2k-1)^2}\right)}{c_1^2} - \frac{1}{c_2} \\ &= \frac{\left[2kc_2 - (2k-1)c_1\right]\left[(2k-1)c_1 - 2(k-1)c_2\right]}{(2k-1)^2 c_1^2 c_2}, \end{aligned}$$

a condition satisfied if and only if

$$\frac{2k}{2k-1}c_2 \geq c_1 \geq \frac{2(k-1)}{2k-1}c_2.$$

If $c_1 = \frac{2(k-1)}{2k-1}c_2$, then the proposed equilibrium strategies have $p_{1,k} = 0$, in which case player 1 does not truly have x_1^{sa} in the support of his strategy. Therefore, the proposed strategies form an equilibrium in which players have k elements in the supports of their strategies if and only if

$$\frac{2k}{2k-1}c_2 \geq c_1 > \frac{2(k-1)}{2k-1}c_2.$$

□

Proof of Proposition 2. First, we complete the discussion preceding Proposition 2 and establish that g in (11) is indeed a symmetric equilibrium strategy. If all players but player 1 use strategy g in (11), then player 1 can implement any level of group effort $g(c')$ in the interval $[0, g(\underline{c})]$ by acting as a type c'. From (8) we see

$$\begin{aligned} \frac{\partial V(c', c)}{\partial c'} &= g'(c')\left[-c - (1 - F^M(c'))v'(g(c'))\right] \\ &= g'(c)\left[-c + c'\right]. \qquad \text{(by (9))} \end{aligned}$$

Because g is decreasing, it follows that the above derivative is positive for $c' < c$ and negative for $c' \in (c, c^*)$. Therefore, player 1 with cost c cannot do better acting as some other type using g. It only remains to verify he cannot do better by choosing some level of effort above $g(\underline{c})$. Choosing $\gamma \geq g(\underline{c})$ yields player 1 the payoff $v(\gamma) - c\gamma$, for which $\frac{\partial}{\partial \gamma}(v(\gamma) - c\gamma) = v'(\gamma) - c \leq v'(\gamma) - \underline{c} \leq 0$ for any $\gamma > g(\underline{c})$. Consequently, no type wishes to exert effort greater than $g(\underline{c})$. Thus, we have established that g in (11) describes a symmetric equilibrium.

Next we show it is the unique symmetric equilibrium. Let g_s be any symmetric equilibrium strategy. By the usual incentive compatibility arguments, g_s is weakly decreasing. Now, note that $g_s(\bar{c}) = 0$, since otherwise type $\bar{c}$ of one player could reduce his effort and save on his costs without changing the expected level of provision of the public good. Also, $g_s(\underline{c}) > 0$, which implies that the no-effort strategy is not a symmetric equilibrium. To see this, suppose to the contrary that $g_s(c) = 0$ for all $c \in (\underline{c}, \bar{c})$. The condition $v'(0) > \underline{c}$ then guarantees a strictly profitable deviation to a positive effort level by type $\underline{c}$.

We now establish that g_s is continuous on $(\underline{c}, \bar{c})$. Suppose to the contrary that a discontinuity at $\tilde{c} \in (\underline{c}, \bar{c})$ exists, so that $\lim_{c\uparrow\tilde{c}} g_s(c) = g_s^h > g_s^l = \lim_{c\downarrow\tilde{c}}$. Thus, for any effort $\gamma \in (g_s^l, g_s^h)$, the utility of type c is

$$V^{nc}(c,\gamma) = -c\gamma + (1 - F^M(\tilde{c}))v(\gamma) + \int_{\underline{c}}^{\tilde{c}} v(g_s(y)) f^M(y)\, dy.$$

Note also that V^{nc} is continuous for $\gamma \in [g_s^l, g_s^h]$, even if there exist a mass of probability in the distribution of efforts at g_s^l or g_s^h. This is because the identity of player with the "best-shot" does not affect the prize, which is a pure public good, nor the payment, which is always required. Hence, since g_s is an equilibrium, the following necessary condition must hold:

$$\frac{\partial V^{nc}}{\partial \gamma}(\tilde{c}, g_s^l) \leq 0 \leq \frac{\partial V^{nc}}{\partial \gamma}(\tilde{c}, g_s^h), \tag{22}$$

reflecting the fact that $\tilde{c}$ does not want to exert effort greater than g_s^l or less than g_s^h. But

$$\frac{\partial V^{nc}}{\partial \gamma}(\tilde{c}, g_s^l) = -\tilde{c} + (1 - F^M(\tilde{c}))v'(g_s^l) > -\tilde{c} + (1 - F^M(\tilde{c}))v'(g_s^h) = \frac{\partial V^{nc}}{\partial \gamma}(\tilde{c}, g_s^h),$$

because v' is decreasing, thus contradicting (22).

Using continuity of g_s, the previously derived bounds, and the fact that g_s must obey equation (9) when g_s is strictly decreasing, we now rule out flat spots at an effort level $\gamma \in (0, g_s(\underline{c}))$. Again, proceeding by contradiction, if such a flat spot exists then

$$c^\gamma \equiv \sup\{c : g_s(c) > \gamma\} < \inf\{c : g_s(c) < \gamma\} \equiv c_\gamma.$$

By continuity of g_s, there exist strictly decreasing segments of g_s with range $(\gamma - \varepsilon, \gamma)$ and $(\gamma, \gamma + \varepsilon)$. Taking limits in (9), this implies $\frac{c^\gamma}{1-F^M(c^\gamma)} = v'(\gamma) = \frac{c_\gamma}{1-F^M(c_\gamma)}$, which contradicts $c^\gamma < c_\gamma$, since the function $\frac{c}{1-F^M(c)}$ is strictly increasing in c.

We now rule out a flat spot at a effort level $\gamma = g_s(\underline{c})$. To the contrary, suppose such flat spot exists. The above discussion implies the existence of a strictly decreasing segment of g_s with range $(\gamma-\varepsilon,\gamma)$. Hence, again letting $c_\gamma=\inf\{c: g(c)<\gamma\}$, we have $\underline{c}<c_\gamma$ and $v'(\gamma)=\frac{c_\gamma}{1-F^M(c_\gamma)}$ by (9), implying $v'(\gamma)>c_\gamma$. For any effort $\gamma'>\gamma$ the utility of type c is $-c\gamma'+v(\gamma')$. Hence, type c_γ has a profitable deviation to an effort level marginally higher than γ, since $v'(\gamma)>c_\gamma$.

Finally, we rule out a flat spot at the effort level of zero, unless the flat spot occurs for types larger than c^*. By contradiction, let there be $\tilde{c}<c^*$ such that $g_s(c)=0$ for all $c>\tilde{c}$. The previous discussion implies the existence of a strictly decreasing segment of g_s with range $(0,\varepsilon)$. Hence, equation (9) implies $v'(0)=\frac{\tilde{c}}{1-F^M(\tilde{c})}$. If $v'(0)=+\infty$ this is impossible. If $v'(0)<\infty$, the definition of c^* and the fact that $\frac{c}{1-F^M(c)}$ is strictly increasing in c imply $\tilde{c}=c^*$, and again we have a contradiction.

Therefore, the only possibility for a symmetric equilibrium is for g_s to equal the formulation in (11). □

Proof of Lemma 2

Recall that here we consider two continuous distributions F_1 and F_2 on $[\underline{c},\bar{c}]$ and we assume F_2 SOSD F_1: $\int_{\underline{c}}^{y}(F_1(c)-F_2(c))\,dc\geq 0$ for all $y\geq\underline{c}$. The following lemma proves very useful in the rest of the analysis.

Lemma 3 (SOSD implications)**.** *Suppose F_2 SOSD F_1, and let $W(c)$ be a non-negative, decreasing function. Then*

$$\int_{\underline{c}}^{y}(F_2(c)-F_1(c))W(c)\,dc\leq 0.$$

Proof of Lemma 3. Define $\Delta(c)\equiv F_2(c)-F_1(c)$ and

$$\bar{\Delta}(y)\equiv\int_{\underline{c}}^{y}\Delta(c)\,dc.$$

Then

$$\begin{aligned}\int_{\underline{c}}^{y}\Delta(c)W(c)\,dc &= \bar{\Delta}(c)W(c)\big|_{\underline{c}}^{y}-\int_{\underline{c}}^{y}\bar{\Delta}(c)W'(c)\,dc\\ &= \underbrace{\bar{\Delta}(y)}_{\leq 0\text{ by SOSD}}\underbrace{W(y)}_{\geq 0}-\int_{\underline{c}}^{y}\underbrace{\bar{\Delta}(c)}_{\leq 0}\underbrace{W'(c)}_{\leq 0}\,dc\\ &\leq 0,\end{aligned}$$

where the second equality uses $\bar{\Delta}(\underline{c})=0$. □

We now establish two results about F_1^M and F_2^M. First, note that, since

$$F_i^M(c)=1-(1-F_i(c))^{n-1},\quad i=1,2,$$

we have that

$$F_1^M(c) \geq F_2^M(c) \iff F_1(c) \geq F_2(c).$$

Thus, whether we consider the pair F_1, F_2 or the pair F_1^M, F_2^M, all intersection points and all directions of the intersections are the same. Our second result is that

$$\int_{\underline{c}}^{y} F_1^M(c)dc \geq \int_{\underline{c}}^{y} F_2^M(c)dc. \tag{23}$$

To see this, note that (23) is equivalent to

$$\int_{\underline{c}}^{y} \left((1 - F_1(c))^{n-1} - (1 - F_2(c))^{n-1}\right) dc \leq 0,$$

or, decomposing the difference of powers,

$$\int_{\underline{c}}^{y} (F_2(c) - F_1(c))P(c)dc \leq 0, \tag{24}$$

where

$$P(c) = \sum_{j=1}^{n-1} (1 - F_1(c))^{n-1-j} (1 - F_2(c))^{j-1}$$

is positive and decreasing in c. Therefore, Lemma 3 applies and (24) is true, which in turn establishes (23).

Denote now with n_{int} the number of strict intersections between F_2 and F_1, not including the those that may occur at $\underline{c}$ and $\bar{c}$. To reflect SOSD of F_2 over F_1, F_2 must overtake F_1 at the first strict intersection point. We label this point as c^1. It then follows that F_2 overtakes F_1 at all odd-numbered strict intersection points. The situation we envision is depicted in Figure 2. (n_{int} is odd there, but this is not essential.)

Proof of Lemma 2. We conduct the proof for the case $n_{int} = 3$, as illustrated in Figure 2, but the reasoning easily generalizes to any n_{int}. (Footnote 17, *infra*, outlines the simple changes in the proof that are required when $n_{int} > 3$.) The conclusion we want to prove is that for any y we have

$$\int_{\underline{c}}^{y} (g(c|F_1) - g(c|F_2))\, dc \leq 0. \tag{25}$$

Denote as Case 1 the situation in which y is so that $F_1(y) > F_2(y)$. Case 2 then occurs when $F_1(y) \leq F_2(y)$. We explicitly demonstrate our conclusion only for $y > c^3$, a sub-case of Case 1. This is because, as the following proof makes clear, other values of y make establishing (25) easier. (We briefly comment on Case 2 at the end of this proof.)

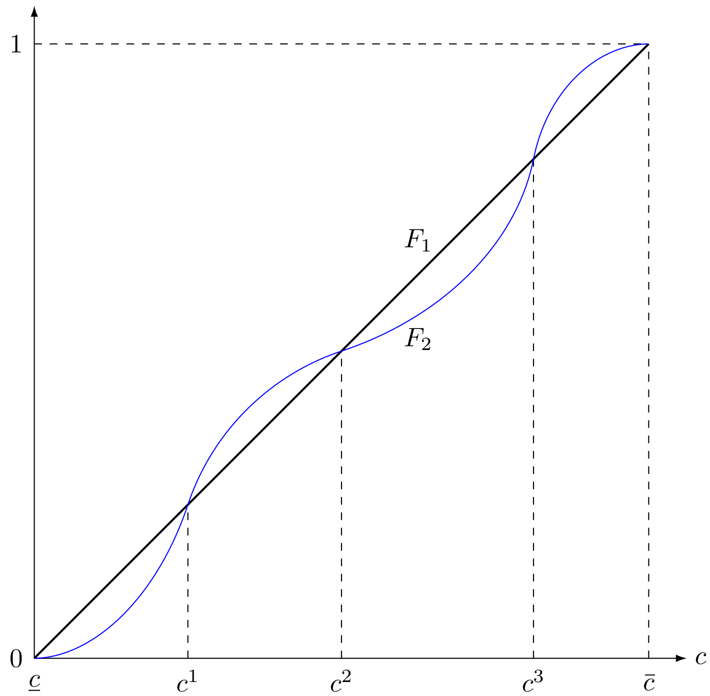

Figure 2: Two cdfs ordered by SOSD, crossing thrice.

We begin by establishing that $\frac{\partial^2 \psi}{\partial z^2} \geq 0$ implies $\frac{\partial^2 \psi}{\partial z \partial c} \geq 0$. Indeed, through the definition of ψ we obtain

$$\frac{\partial^2 \psi(c,z)}{\partial z^2} \cdot v''(\psi(c,z)) = \frac{c}{(1-z)^3} \left(2 - \frac{v'(\psi(c,z))}{v''(\psi(c,z))} \frac{v'''(\psi(c,z))}{v''(\psi(c,z))} \right), \tag{26}$$

and

$$\frac{\partial^2 \psi(c,z)}{\partial z \, \partial c} \cdot v''(\psi(c,z)) = \frac{1}{(1-z)^2} \left(1 - \frac{v'(\psi(c,z))}{v''(\psi(c,z))} \frac{v'''(\psi(c,z))}{v''(\psi(c,z))} \right).$$

Therefore,

$$\frac{\partial^2 \psi(c,z)}{\partial z^2} \cdot v''(\psi(c,z)) \cdot \frac{(1-z)^3}{c} \geq \frac{\partial^2 \psi(c,z)}{(\partial z)\,(\partial c)} \cdot v''(\psi(c,z)) \cdot (1-z)^2,$$

so, since $z \leq 1$ and $v'' \leq 0$, if $\frac{\partial^2 \psi}{\partial z^2} \geq 0$, then $\frac{\partial^2 \psi}{\partial z \partial c} \geq 0$. Having established the signs of the partial derivatives, we now proceed to prove the lemma. The integral of interest in (25) can be rewritten as

$$\begin{aligned} \int_{\underline{c}}^{y} \left(g(c|F_1) - g(c|F_2)\right) dc = &\int_{\underline{c}}^{c^1} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\ &+ \int_{c^1}^{c^2} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\ &+ \int_{c^2}^{c^3} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\ &+ \int_{c^3}^{y} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc. \end{aligned} \tag{27}$$

Consider the term $\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)$ in the first integral on the right-hand side of (27). Since $c \leq c^1$, we have $F_2^M(c) \leq F_1^M(c)$, so, using $\frac{\partial^2 \psi}{\partial z^2} \geq 0$ and $w \leq F_1^M(c) \leq F_1^M\left(c^1\right)$, we obtain

$$\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right) = \int_{F_2^M(c)}^{F_1^M(c)} \frac{\partial \psi}{\partial z}(c, w)\, dw \leq \frac{\partial \psi}{\partial z}\left(c, F_1^M\left(c^1\right)\right)\left(F_1^M(c) - F_2^M(c)\right).$$

Now, using $\frac{\partial^2 \psi}{\partial z \partial c} \geq 0$, we see that the first term on the right-hand side of (27) can be bounded above as

$$\int_{\underline{c}}^{c^1} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \leq \frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right) \int_{\underline{c}}^{c^1} \left(F_1^M(c) - F_2^M(c)\right) dc. \tag{28}$$

Consider now the second integral on the right-hand side of (27). Since $c^1 \leq c \leq c^2$, we have $F_2^M(c) > F_1^M(c)$, so

$$\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right) = \int_{F_2^M(c)}^{F_1^M(c)} \frac{\partial \psi}{\partial z}(c, w)\, dw = \int_{F_1^M(c)}^{F_2^M(c)} \left(-\frac{\partial \psi}{\partial z}(c, w)\right) dw.$$

Since $\left(-\frac{\partial^2 \psi}{\partial z^2}\right) \leq 0$ and $w \geq F_1^M(c) \geq F_1^M\left(c^1\right)$, we have

$$\int_{F_1^M(c)}^{F_2^M(c)} \left(-\frac{\partial \psi}{\partial z}(c, w)\right) dw \leq \left(-\frac{\partial \psi}{\partial z}\left(c, F_1^M\left(c^1\right)\right)\right)\left(F_2^M(c) - F_1^M(c)\right),$$

and since $\left(-\frac{\partial^2 \psi}{\partial z \partial c}\right) \leq 0$ and $c \geq c^1$, we obtain

$$\begin{aligned}\int_{F_1^M(c)}^{F_2^M(c)} \left(-\frac{\partial \psi}{\partial z}(c, w)\right) dw &\leq \left(-\frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right)\right)\left(F_2^M(c) - F_1^M(c)\right) \\ &= \left(\frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right)\right)\left(F_1^M(c) - F_2^M(c)\right).\end{aligned}$$

All in all, therefore, the second term on the right-hand side of (27) can be bounded above as

$$\int_{c^1}^{c^2} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \leq \frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right) \int_{c^1}^{c^2} \left(F_1^M(c) - F_2^M(c)\right) dc. \tag{29}$$

Therefore, from (28) and (29) we see the sum of the first and second terms on the right-hand side of (27) is bounded above by

$$\frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right) \int_{\underline{c}}^{c^2} \left(F_1^M(c) - F_2^M(c)\right) dc. \tag{30}$$

Proceeding in a similar fashion, the sum of the third and fourth terms on the right-hand side of (27) is bounded above by

$$\frac{\partial \psi}{\partial z}\left(c^3, F_1^M\left(c^3\right)\right) \int_{c^2}^{y} \left(F_1^M(c) - F_2^M(c)\right) dc. \tag{31}$$

Note now that, by (23), the integral in (30) is non-negative. Therefore, using $\frac{\partial^2\psi}{\partial z^2} \geq 0$ and $\frac{\partial^2\psi}{\partial z \partial c} \geq 0$, the value in (30) is in turn bounded above by[17]

$$\frac{\partial \psi}{\partial z}\left(c^3, F_1^M\left(c^3\right)\right) \int_{\underline{c}}^{c^2} \left(F_1^M(c) - F_2^M(c)\right) dc.$$

Thus, using the above displayed value and equation (31), equation (27) is bounded above as

$$\int_{\underline{c}}^{y} \left(g(c|F_1) - g(c|F_2)\right) dc \leq \left(\frac{\partial \psi}{\partial z}\left(c^3, F_1^M\left(c^3\right)\right)\right) \int_{\underline{c}}^{y} \left(F_1^M(c) - F_2^M(c)\right) dc.$$

Note that the first term in the right-hand side of the above is negative, by concavity of v. And the second term is positive, by (23). Hence

$$\int_{\underline{c}}^{y} g(c|F_1) dc \leq \int_{\underline{c}}^{y} g(c|F_2) dc,$$

as we wanted to show to establish (25).

We now briefly discuss Case 2. With reference to Figure 2, take $y \in [c_2, c_3]$. (The argument easily adapts to other values of y belonging to Case 2.) The integral of interest in (25) can be rewritten as

$$\begin{aligned}
\int_{\underline{c}}^{y} \left(g(c|F_1) - g(c|F_2)\right) dc = & \int_{\underline{c}}^{c^1} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\
& + \int_{c^1}^{c^2} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\
& + \int_{c^2}^{y} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\
& \leq \frac{\partial \psi}{\partial z}\left(c^1, F_1^M\left(c^1\right)\right) \int_{\underline{c}}^{c^2} \left(F_1^M(c) - F_2^M(c)\right) dc && \text{by (30)} \\
& + \int_{c^2}^{y} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc \\
& \leq \int_{c^2}^{y} \left(\psi\left(c, F_1^M(c)\right) - \psi\left(c, F_2^M(c)\right)\right) dc. && \text{by (23) and } \frac{\partial \psi}{\partial z} \leq 0
\end{aligned}$$

Therefore, using the facts that $\frac{\partial \psi}{\partial z} \leq 0$ and $F_1^M(c) > F_2^M(c)$ for $c \in [c_2, y]$ (by $y \in [c_2, c_3]$), we have again established (25). □

Proof that the strategies g_1 and g_2 of Example 5 form an equilibrium. We begin by establishing that g_1 is optimal given g_2. First note that for efforts γ larger than $\hat{\gamma}$ the appropriate expression for payoffs is $V(\gamma, c_1)$ in (14), with derivative

$$\frac{\partial V(\gamma, c_1)}{\partial \gamma} = -c_1 + v'(\gamma)(1 - F_2(\phi_2(\gamma))).$$

[17] If n_{int} were larger than 3, then this step would have to be repeated up to $c^{n_{int}}$, not just up to c^3.

Hence, for $c_1 \leq \hat{c}_1$, g_1 solves the necessary first-order condition for optimality, since ϕ_1^H solves (15). For $c_1 > \hat{c}_1$, g_1 prescribes efforts smaller than $\hat{\gamma}$. In this case, the appropriate expression for payoffs is

$$\tilde{V}(\gamma, c_1) = -c_1\gamma + \int_{\underline{c}}^{\hat{c}_2} v(g_2(c_2)) f_2(c_2) dc_2 + (1 - F_2(\hat{c}_2)) v(\gamma),$$

with derivative

$$\frac{\partial \tilde{V}(\gamma, c_1)}{\partial \gamma} = -c_1 + v'(\gamma)(1 - F_2(\hat{c}_2)).$$

As it can easily be checked substituting the functional forms for v and F_2 in this example, also for this range of costs and efforts, g_1 solves the necessary condition for optimality, which now reads as

$$c_1 = v'(g_1(c_1))(1 - F_2(\hat{c}_2)). \tag{32}$$

We now establish that these necessary conditions are also sufficient. Take $c_1 < \hat{c}_1$ and consider a deviation to $\gamma^D < \hat{\gamma}$. The difference in utility brought about by the deviation is then

$$\tilde{V}(\gamma^D, c_1) - V(g_1(c_1), c_1) = -\left[\int_{\gamma^D}^{\hat{\gamma}} \frac{\partial \tilde{V}}{\partial \gamma}(y, c_1)\, dy + \int_{\hat{\gamma}}^{g_1(c_1)} \frac{\partial V}{\partial \gamma}(y, c_1)\, dy\right].$$

Now, for $y > \hat{\gamma}$, we have

$$\frac{\partial V}{\partial \gamma}(y, c_1) = -c_1 + v'(y)(1 - F_2(\phi_2(y))) = -c_1 + \phi_1^H(y),$$

by (15) and the formulation of g_1 in this example. And since $y < g_1(c_1)$, we obtain $\partial V / \partial \gamma > 0$. Similarly, for $y < \hat{\gamma}$, we have

$$\begin{aligned}
\frac{\partial \tilde{V}}{\partial \gamma}(y, c_1) &= -c_1 + v'(y)(1 - F_2(\hat{c}_2)) \\
&= -c_1 + \phi_1(y) && \text{(by (32))} \\
&> 0, && \text{(since } y < \hat{\gamma} < g_1(c_1)\text{)}.
\end{aligned}$$

Therefore, $\tilde{V}(\gamma^D, c_1) - V(g_1(c_1), c_1) < 0$ and the deviation is strictly unprofitable. Similar considerations establish the optimality of $g_1(c_1)$ for $c_1 > \hat{c}_1$ and for deviations to $\gamma^D > \hat{\gamma}$.

Now, turning our attention to player 2, one can establish the optimality of $g_2(c_2)$ for $c_2 < \hat{c}_2$ as above; differences in the proof arise only for $c_2 \geq \hat{c}_2$. Type $\hat{c}_2$ must be indifferent between efforts 0 and $\hat{\gamma}$. Thus, the condition to be satisfied is

$$-\hat{c}_2\hat{\gamma} + (1 - F_1(\hat{c}_1))v(\hat{\gamma}) + \int_{\underline{c}}^{\hat{c}_1} v(g_1(c_1))f_1(c_1)\,dc_1 = \int_{\underline{c}}^{\bar{c}} v(g_1(c_1))f_1(c_1)\,dc_1,$$

which, for $v = 2\sqrt{x}$ and our g_1, reduces to (16). Note also that if $\hat{c}_2$ weakly prefers 0 to $\hat{\gamma}$, then all types $c_2 > \hat{c}_2$ strictly prefer to exert no effort. And this is true not just for effort $\hat{\gamma}$, but for any effort larger than zero. Thus, the proof is complete if we can show that $\hat{c}_2$ weakly prefers 0 to $\gamma \in (g_1(\bar{c}), \hat{\gamma})$. The utility of such an effort is

$$V^D(\gamma, \hat{c}_2) = -\hat{c}_2\gamma + \int_{\underline{c}}^{\phi_1(\gamma)} v(g_1(c_1))\,dF_1 + (1 - F_1(\phi_1(\gamma)))v(\gamma),$$

with derivative with respect to γ equal to

$$-\hat{c}_2 + (1 - F_1(\phi_1(\gamma)))v'(\gamma).$$

We now use this derivative to show that V^D has no interior maximum. Substituting the functional forms for this example, the above derivative is equal to

$$\begin{aligned}
&-\hat{c}_2 + \frac{10}{9}\left(1 - \frac{100}{99}\frac{1-(\hat{c}_2)^2}{\sqrt{\gamma}}\right)\frac{1}{\sqrt{\gamma}} \\
&= \frac{-1000 + 1000(\hat{c}_2)^2 + 990\sqrt{\gamma} - 891(\hat{c}_2)\gamma}{891\gamma} \\
&= \frac{-(\hat{c}_2)}{\gamma}\left(\sqrt{\gamma} - \frac{5(11 - \sqrt{11}\sqrt{11 - 40\hat{c}_2 + 40(\hat{c}_2)^3})}{99\hat{c}_2}\right)\left(\sqrt{\gamma} - \frac{5(11 + \sqrt{11}\sqrt{11 - 40\hat{c}_2 + 40(\hat{c}_2)^3})}{99\hat{c}_2}\right) \\
&= \frac{-(\hat{c}_2)}{\gamma}\left(\sqrt{\gamma} - \frac{5(11 - \sqrt{11}\sqrt{11 - 40\hat{c}_2 + 40(\hat{c}_2)^3})}{99\hat{c}_2}\right)\left(\sqrt{\gamma} - \sqrt{\hat{\gamma}}\right),
\end{aligned}$$

where the last equality uses $\hat{c}_2 = \phi_2^H(\hat{\gamma})$. Hence, $V^D(\gamma, \hat{c}_2)$ has no interior maximum for $\gamma \in (g_1(\bar{c}), \hat{\gamma})$: the only critical point is

$$\gamma = \left(\frac{5(11 - \sqrt{11}\sqrt{11 - 40\hat{c}_2 + 40(\hat{c}_2)^3})}{99\hat{c}_2}\right)^2,$$

and it is a minimum. Now, note that $\gamma = g_1(\bar{c})$ is clearly worse than taking no effort. Indeed, player 2 cannot benefit from this deviation because the public good is produced in exactly the same quantity as if 2's effort were zero, but now player 2 pays the cost of a strictly positive effort. Finally, note that the utility of effort $\hat{\gamma}$ is the same obtained with zero effort, by the indifference condition (16). Thus, type $\hat{c}_2$ does not want to deviate and the proof is complete. □

Proof of Proposition 4. We first prove part (i). Observe that $g_k(y) = x^{\text{sa}}\left(\frac{y}{(1-F(y))^{k-1}}\right)$ for any $k \geq 1$. Fix $n \geq 2$. Because $g_n(y) \leq g_2(y)$ for all y, we have $x^{\text{sa}}(y)(1-F(y)) - g_n(y) \geq x^{\text{sa}}(y)(1-F(y)) - g_2(y)$, so to show (18) is strictly positive, it suffices to show

$$x^{\text{sa}}(y)(1-F(y)) - g_2(y) = x^{\text{sa}}(y)(1-F(y)) - x^{\text{sa}}\left(\frac{y}{1-F(y)}\right) \geq 0, \tag{33}$$

with strict inequality on $y \in (\underline{c}, c_2^*)$. On $(c_2^*, \bar{c}]$, (33) is necessarily nonnegative as here $g_2(y) = 0$. By assumption of part (i), $sx^{\text{sa}}(s)$ is decreasing in s, so for any $y \in (\underline{c}, c_2^*]$ we have $y < y/(1-F(y)$, implying

$$yx^{\text{sa}}(y) > \frac{y}{(1-F(y)} \; x^{\text{sa}}\left(\frac{y}{1-F(y)}\right),$$

which in turn implies $x^{\text{sa}}(y)(1-F(y)) > x^{\text{sa}}\left(\frac{y}{1-F(y)}\right)$; that is, (33) is satisfied.

Next consider part (ii). We show that for all n sufficiently large,

$$x^{\text{sa}}(y)(1-F(y)) - g_n(y) \geq 0, \quad \text{with strict inequality on } (\underline{c}, c_1^*). \tag{34}$$

First we consider a neighborhood of $\underline{c}$. We see at $y = \underline{c}$ both $x^{\text{sa}}(y)(1-F(y))$ and $g_n(y)$ equal $x^{\text{sa}}(\underline{c}) > 0$, where the inequality follows from $v'(0) > \underline{c}$. Furthermore,

$$\left.\frac{d\left[x^{\text{sa}}(y)(1-F(y))\right]}{dy}\right|_{y=\underline{c}} = \frac{dx^{\text{sa}}(\underline{c})}{dy} - f(\underline{c})x^{\text{sa}}(\underline{c}) \tag{35}$$

and

$$g_n'(\underline{c}) = \frac{dx^{\text{sa}}(\underline{c})}{dy}[1 + (n-1)\underline{c}f(\underline{c})]. \tag{36}$$

Because $dx^{\text{sa}}(\underline{c})/dy$ is negative and finite, $x^{\text{sa}}(\underline{c})$ is finite, and $f(\underline{c}) > 0$, it now follows from (35) and (36) that there exists n' such that the expression in (36) is strictly less than that in (35) for all $n \geq n'$. Consequently, there exists $\varepsilon > 0$ such that

$$x^{\text{sa}}(y)(1-F(y)) - g_{n'}(y) > 0 \quad \forall y \in (\underline{c}, \underline{c} + \varepsilon). \tag{37}$$

Because $v'(0) < \infty$, it follows from (10) that $(c_k^*)_k$ is a strictly decreasing sequence with limit $\underline{c}$. So fix

$n'' \geq n'$ such that $c^*_{n''} < \underline{c} + \varepsilon$. Then

$$x^{\mathrm{sa}}(y)(1 - F(y)) - g_{n''}(y) = x^{\mathrm{sa}}(y)(1 - F(y)) > 0 \quad \forall y \in [\underline{c} + \varepsilon, c^*_1) \tag{38}$$

Because the strategies $(g_k)_k$ are pointwise weakly decreasing in n, inequalities (37) and (38) establish the validity of (34) for all $n \geq n''$.

Finally, consider part (iii). Here too we show that for all n sufficiently large, (34) is satisfied. Repeating the arguments of part (ii) we conclude there exists $n' \geq 3$ and $\varepsilon > 0$ such that

$$x^{\mathrm{sa}}(y)(1 - F(y)) - g_{n'}(y) > 0 \quad \forall y \in (\underline{c}, \underline{c} + \varepsilon). \tag{39}$$

Next we consider a neighborhood of $\bar{c}$. (In part (iii), $c^*_k = \bar{c}$ for all k.) Both $x^{\mathrm{sa}}(y)(1 - F(y))$ and $g_{n'}(y)$ go to 0 as $y \to \bar{c}$, and, as we show, of these, $g_{n'}(y)$ goes to 0 much faster. In particular,

$$\begin{aligned}
0 \leq \limsup_{y \to \bar{c}} \frac{g_{n'}(y)}{x^{\mathrm{sa}}(y)(1 - F(y))} &= \limsup_{y \to \bar{c}} \frac{x^{\mathrm{sa}}\left(\frac{y}{(1-F(y))^{n'-1}}\right)}{x^{\mathrm{sa}}(y)(1 - F(y))} \\
&= \limsup_{y \to \bar{c}} \frac{(1 - F(y))^{n'-1}}{y} \times \frac{y}{(1 - F(y))^{n'-1}} \times \frac{x^{\mathrm{sa}}\left(\frac{y}{(1-F(y))^{n'-1}}\right)}{x^{\mathrm{sa}}(y)(1 - F(y))} \\
&= \limsup_{y \to \bar{c}} \left(\frac{(1 - F(y))^{n'-2}}{x^{\mathrm{sa}}(y) y}\right) \underbrace{\left(\frac{y}{(1 - F(y))^{n'-1}}\right) x^{\mathrm{sa}}\left(\frac{y}{(1 - F(y))^{n'-1}}\right)}_{\text{by assumption bounded as } y \to \bar{c}} \\
&= 0
\end{aligned}$$

because $x^{\mathrm{sa}}(\bar{c}) > 0$ (since $v'(0) = \infty$ and $\bar{c} < \infty$) and $n' > 2$. Therefore, there exists $\delta > 0$ such that

$$x^{\mathrm{sa}}(y)(1 - F(y)) - g_{n'}(y) > 0 \quad \forall y \in [\bar{c} - \delta, \bar{c}). \tag{40}$$

Because $\lim_k g_k(y) = 0$ for any $y \in (\underline{c}, \bar{c}]$, we can choose $n'' \geq n'$ be such that $g_{n''}(\underline{c} + \varepsilon) < x^{\mathrm{sa}}(\bar{c} - \delta)(1 - F(\bar{c} - \delta))$, which, because equilibrium strategies are nonincreasing, in turn implies

$$x^{\mathrm{sa}}(y)(1 - F(y)) - g_{n''}(y) > 0 \quad \forall y \in [\underline{c} + \varepsilon, \bar{c} - \delta]. \tag{41}$$

We conclude from (39), (40), and (41) that (34) is satisfied for $n = n''$, and, again because $(g_k)_k$ is pointwise decreasing with respect to k, (34) is satisfied for all $n \geq n''$. $\square$

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